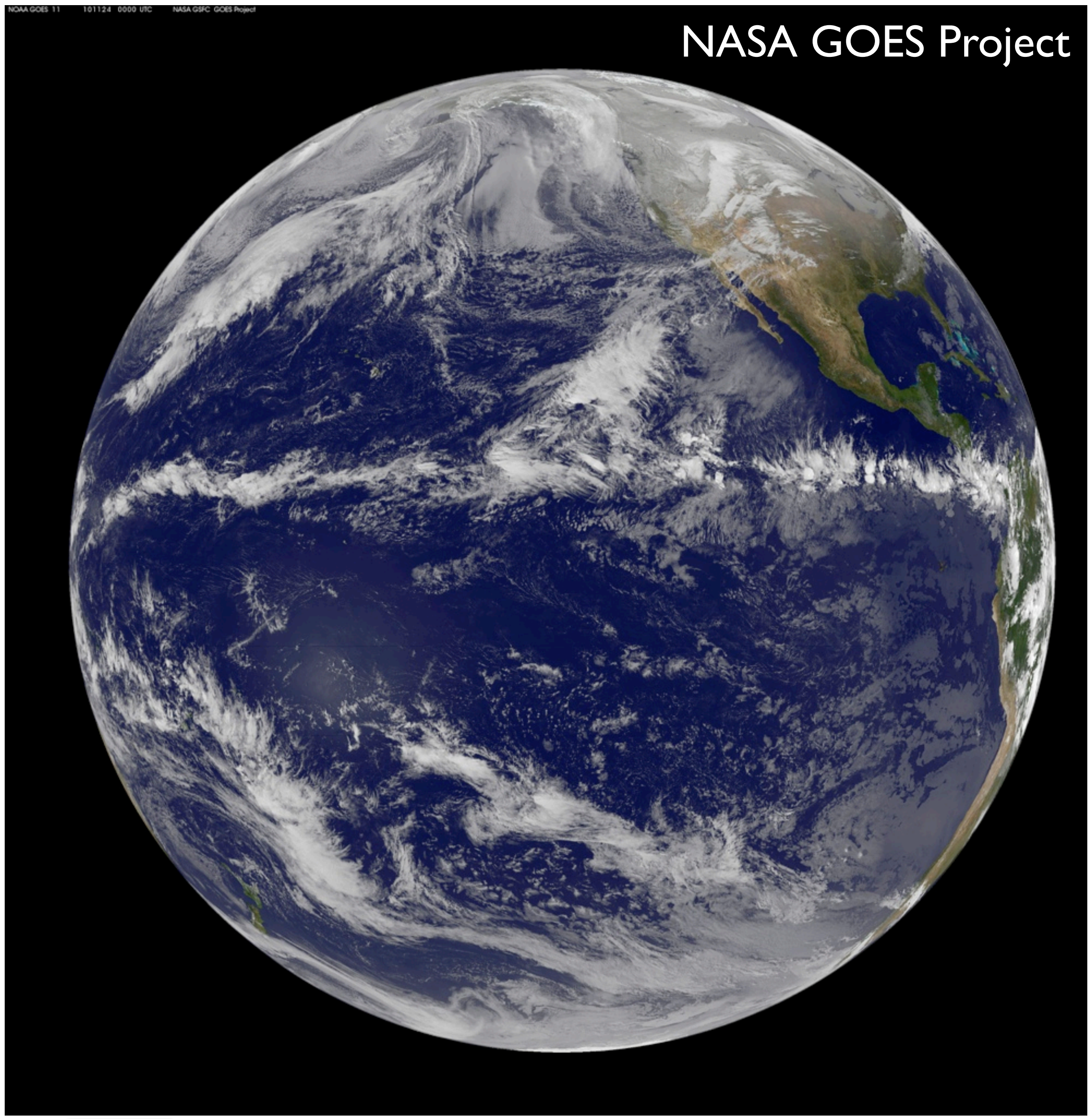
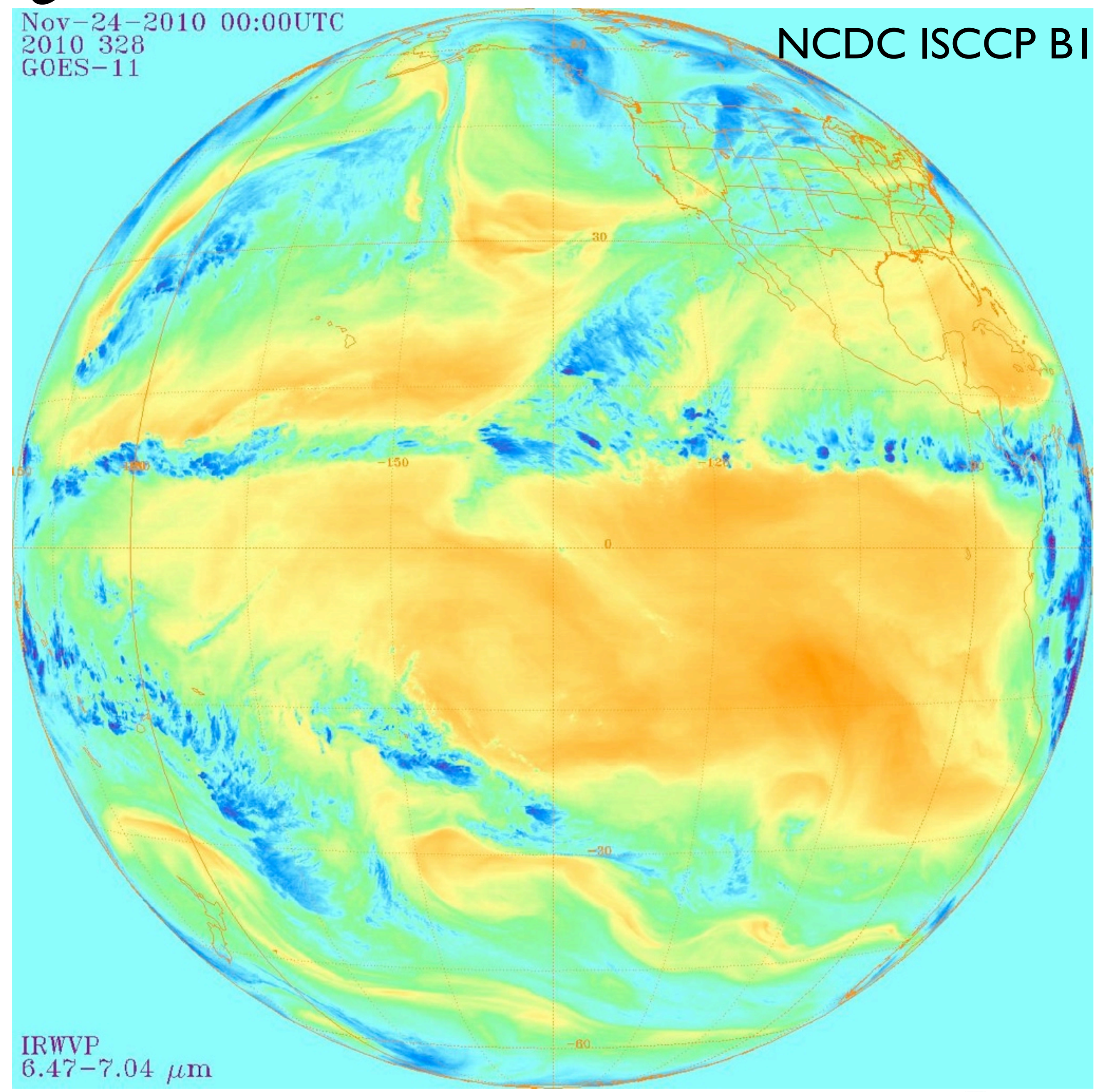
Dynamical aspects of deep and shallow Hadley circulations





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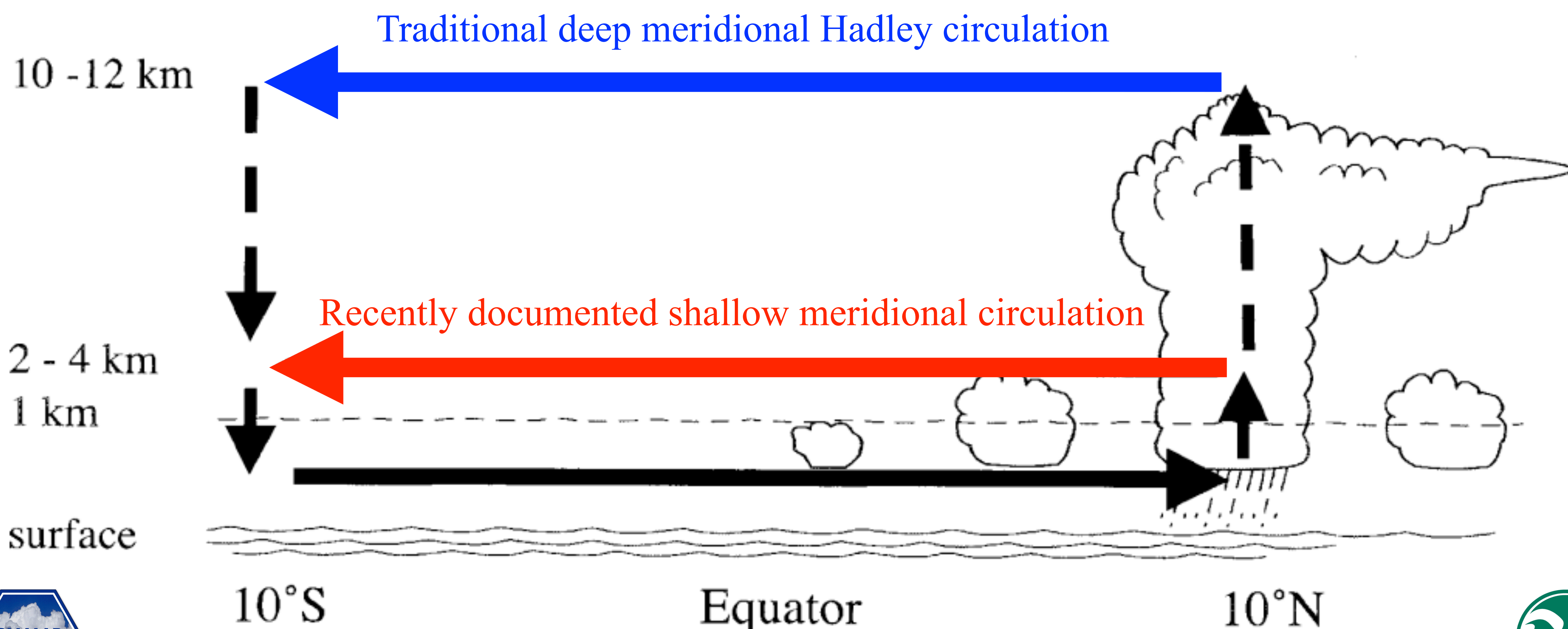
Mean meridional circulations

- **Goal:** Devise an idealized model that improves understanding of dynamics of deep and shallow circulations.

- One set of equations for both Hadley circulations (HCs) that helps describe asymmetries between winter and summer cells.



Adapted from Zhang et al. (2004).

Governing equations

- Zonally symmetric motions linearized about a resting basic state on the equatorial β-plane.

- Log-pressure height coordinate, $0 \le z \le z_T$, of only the inviscid interior.

$$\frac{\partial u}{\partial t} - \beta y v = 0,$$	Zonal momentum equ.
$$\frac{\partial v}{\partial t} + \beta y u + \frac{\partial \phi}{\partial y} = 0,$$	Meridional momentum equ.
$$\frac{\partial \phi}{\partial z} = \frac{g}{T_0} T,$$	Hydrostatic balance
$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} - \frac{w}{H} = 0,$$	Continuity equ.
$$\frac{\partial T}{\partial t} + \frac{T_0}{g} N^2 w = \frac{Q}{c_p},$$	Thermodynamic equ.

Streamfunction PDE

- Solve for the streamfunction, ψ.

$$e^{-z/H} v = -\frac{\partial \psi}{\partial z} \quad \text{and} \quad e^{-z/H} w = \frac{\partial \psi}{\partial y}.$$

- 2nd order hyperbolic PDE with prescribed diabatic heating.

- Shaping parameters are: static stability and inertial stability.

$$N^2 e^{z/H} \frac{\partial^2 \psi}{\partial y^2} + \left(\frac{\partial^2}{\partial t^2} + \beta^2 y^2 \right) \frac{\partial}{\partial z} \left(e^{z/H} \frac{\partial \psi}{\partial z} \right) = \frac{g}{c_p T_0} \frac{\partial Q}{\partial y}$$

with the four boundary conditions (including prescribed Ekman pumping):

$$\psi \to 0 \quad \text{as} \quad y \to \pm\infty,$$

$$\psi = 0 \quad \text{at} \quad z = z_T,$$

$$g \frac{\partial^2 \psi}{\partial y^2} + \left(\frac{\partial^2}{\partial t^2} + \beta^2 y^2 \right) \frac{\partial \psi}{\partial z} = g \frac{\partial W}{\partial y} \quad \text{at} \quad z = 0.$$

CMMAP
Reach for the sky.

Diagnostic equation

- Solve for the streamfunction, ψ.

$$e^{-z/H} v = -\frac{\partial \psi}{\partial z} \quad \text{and} \quad e^{-z/H} w = \frac{\partial \psi}{\partial y}.$$

- Hyperbolic → Elliptic PDE with prescribed diabatic heating.

- Shaping parameters are: static stability and inertial stability.

$$N^2 e^{z/H} \frac{\partial^2 \psi}{\partial y^2} + \left(\cancel{\frac{\partial^2}{\partial t^2}} + \beta^2 y^2 \right) \frac{\partial}{\partial z} \left(e^{z/H} \frac{\partial \psi}{\partial z} \right) = \frac{g}{c_p T_0} \frac{\partial Q}{\partial y}$$

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Vertical transform

$$\hat{\psi}(y,z,t) = \psi(y,z,t)\, e^{z/2H},$$
$$\hat{Q}(y,z,t) = Q(y,z,t)\, e^{-z/2H}.$$

- **Step 1:** Eliminate the z derivatives by performing a vertical transform.

- Similar to a Fourier transform pair, except the domain is not periodic.

$$\hat{\psi}(y,z,t) = \sum_{m=0}^{\infty} \hat{\psi}_m(y,t)\, \mathcal{Z}_m(z)$$

$$\hat{\psi}_m(y,t) = \frac{1}{g}\int_0^{z_T} \hat{\psi}(y,z,t)\, \mathcal{Z}_m(z) N^2(z) dz + \hat{\psi}(y,0,t)\mathcal{Z}_m(0)$$

- Multiply elliptic equation by the eigenfunctions and integrate in z.

- Sum over all vertical wavenumbers, m, multiplied by transform coefficients and eigenfunctions.

Sturm-Liouville Eigenproblem

- The eigenfunctions and associated eigenvalues arise from separating meridional and vertical structure from the elliptic equation for $\hat{\psi}$ and BCs.

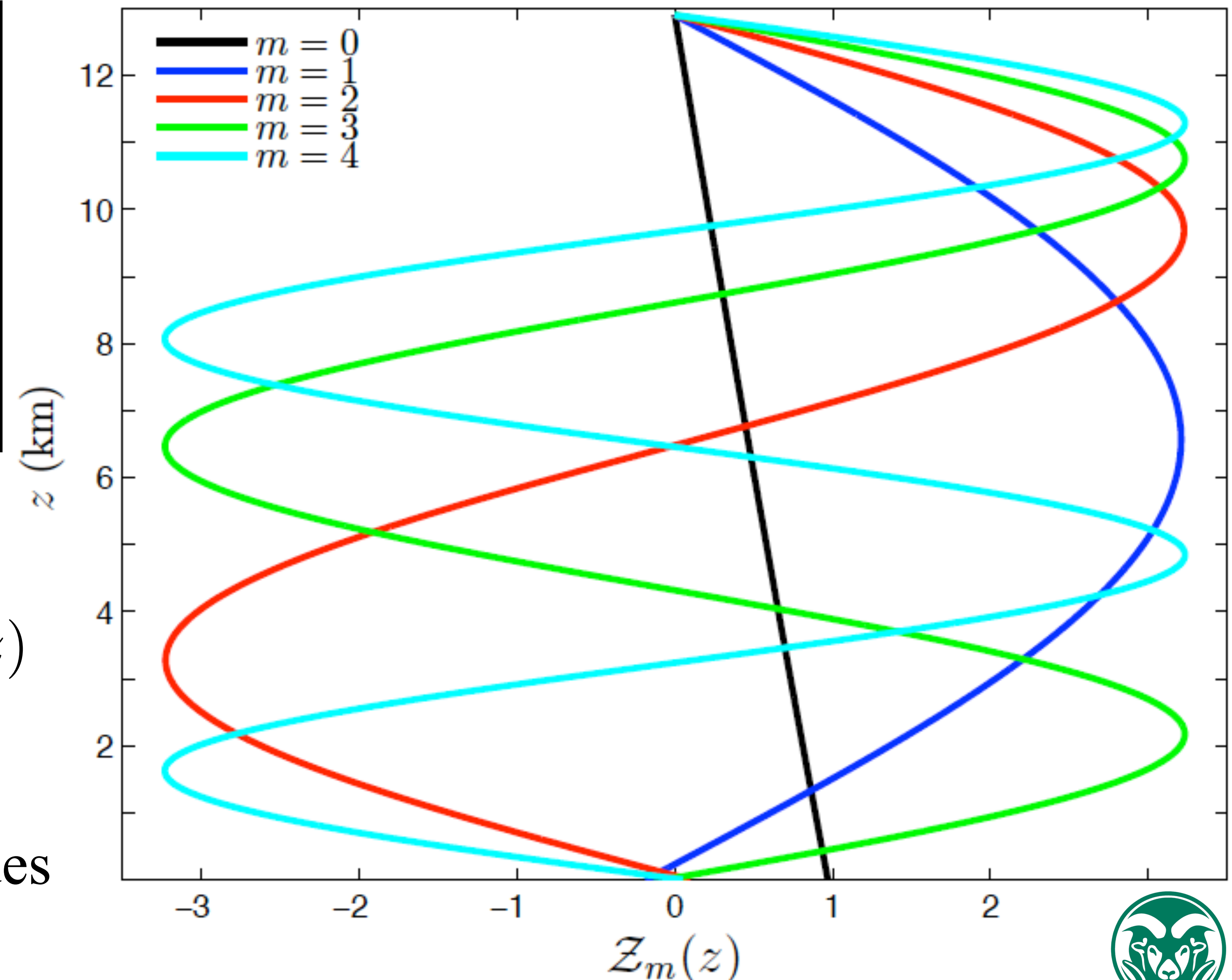
$$\frac{d^2\mathcal{Z}_m}{dz^2} - \frac{\mathcal{Z}_m}{4H^2} = -\frac{N^2\mathcal{Z}_m}{gh_m},$$

$$\mathcal{Z}_m = 0 \text{ at } z = z_T,$$

$$\frac{d\mathcal{Z}_m}{dz} - \frac{\mathcal{Z}_m}{2H} = -\frac{\mathcal{Z}_m}{h_m} \text{ at } z = 0,$$

$$\mathcal{L}\{\mathcal{Z}_m(z)\} = -\frac{N^2}{gh_m}\mathcal{Z}_m(z)$$

Eigenvalues



Meridional equations

- **Step 2:** Eliminate the *y* derivatives by using Green's functions.

- Set of meridional equations where the inertial stability is the only shaping parameter.

$$\frac{\partial^2 \hat{\psi}_m(y,t)}{\partial y^2} - \frac{y^2}{4b_m^4}\hat{\psi}_m(y,t) = \frac{\partial F_m(y,t)}{\partial y},$$

with the two boundary conditions:

$$\hat{\psi}_m(y,t) \to 0 \text{ as } y \to \pm\infty,$$

$$F_m(y,t) = \int_0^{z_T} \frac{\hat{Q}(y,z,t)}{c_p T_0} \mathcal{Z}_m(z)\, dz + \mathcal{W}(y,t)\mathcal{Z}_m(0)$$

Green's functions

- What are Green's functions and why use them? $\mathcal{L}[G(x,s)] = -\delta(x-s)$

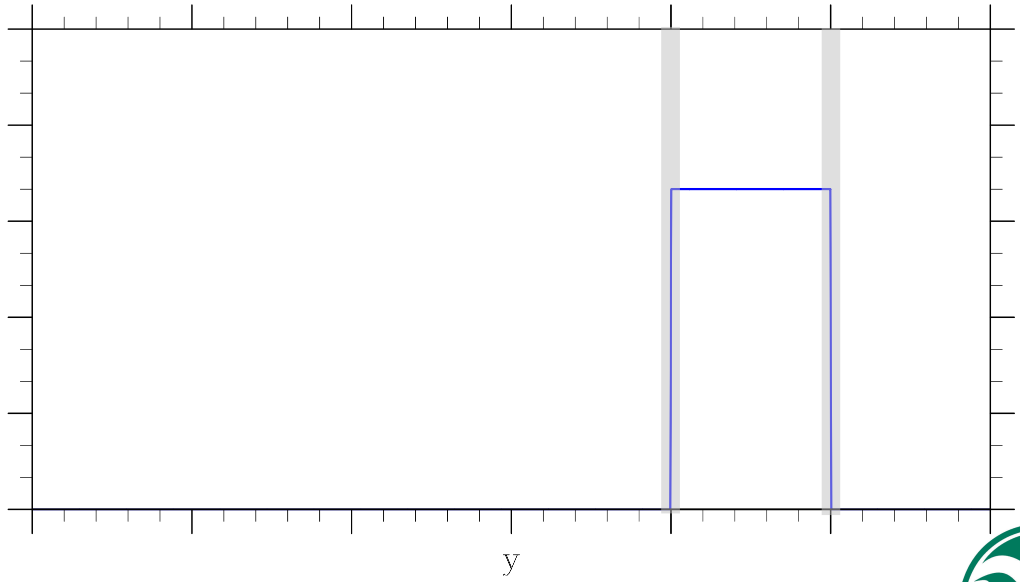
  - We can construct same equations for $G_m(y,y')$ as those for the streamfunction, with same boundary conditions.

$$\frac{d^2 G_m}{dy^2} - \frac{y^2}{4b_m^4} G_m = -\frac{1}{b_m^2}\delta\left(\frac{y-y'}{b_m}\right),$$

with BCs: $G_m(y,y') \to 0 \text{ as } y \to \pm\infty,$

- Green's functions require that RHS be a Dirac delta function.

- This is convenient for forcings that are "top-hat" functions in y.



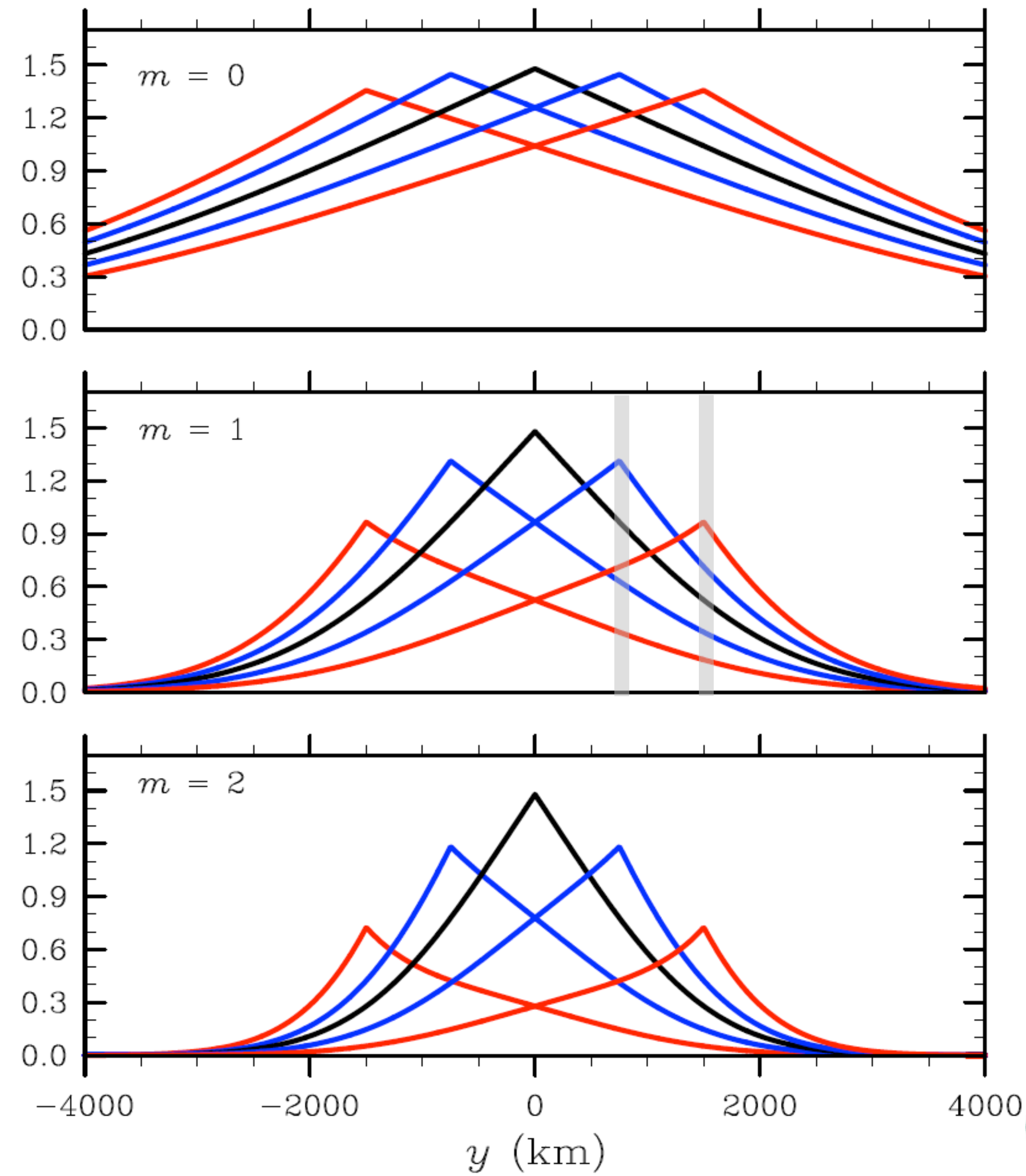
$$G_m(y,y') = \frac{1}{\sqrt{2}} \begin{cases} D_{-1/2}(y'/b_m)D_{-1/2}(-y/b_m) & \text{if } -\infty < y \le y' \\ D_{-1/2}(-y'/b_m)D_{-1/2}(y/b_m) & \text{if } y' \le y < \infty. \end{cases}$$

- $D_{-1/2}(x)$ are parabolic cylinder functions.

- They contain all the information about meridional asymmetries via inertial stability.

- Think about the kinks, y', as the edges of the ITCZ.

$$y' = \pm\, 0, 750, 1500 \text{ km}$$



Final solution

- After combining Green's function equations with equations for $\hat{\psi}_m$, integrate in y.

- The final solution is a sum over all vertical wavenumbers.

$$\psi(y,z,t) = e^{-z/2H} \sum_{m=0}^{\infty} \hat{\psi}_m(y,t)\, \mathcal{Z}_m(z),$$

$$\hat{\psi}_m(y,t) = -b_m \int_{-\infty}^{\infty} \frac{\partial F_m(y',t)}{\partial y'} G_m(y,y')\, dy'.$$

- Forcing #1: Diabatic heating ($m = 1$ only).

- Forcing #2: Ekman pumping at the top of the boundary layer ($z = 0$).

Forcing #1 - Deep HC

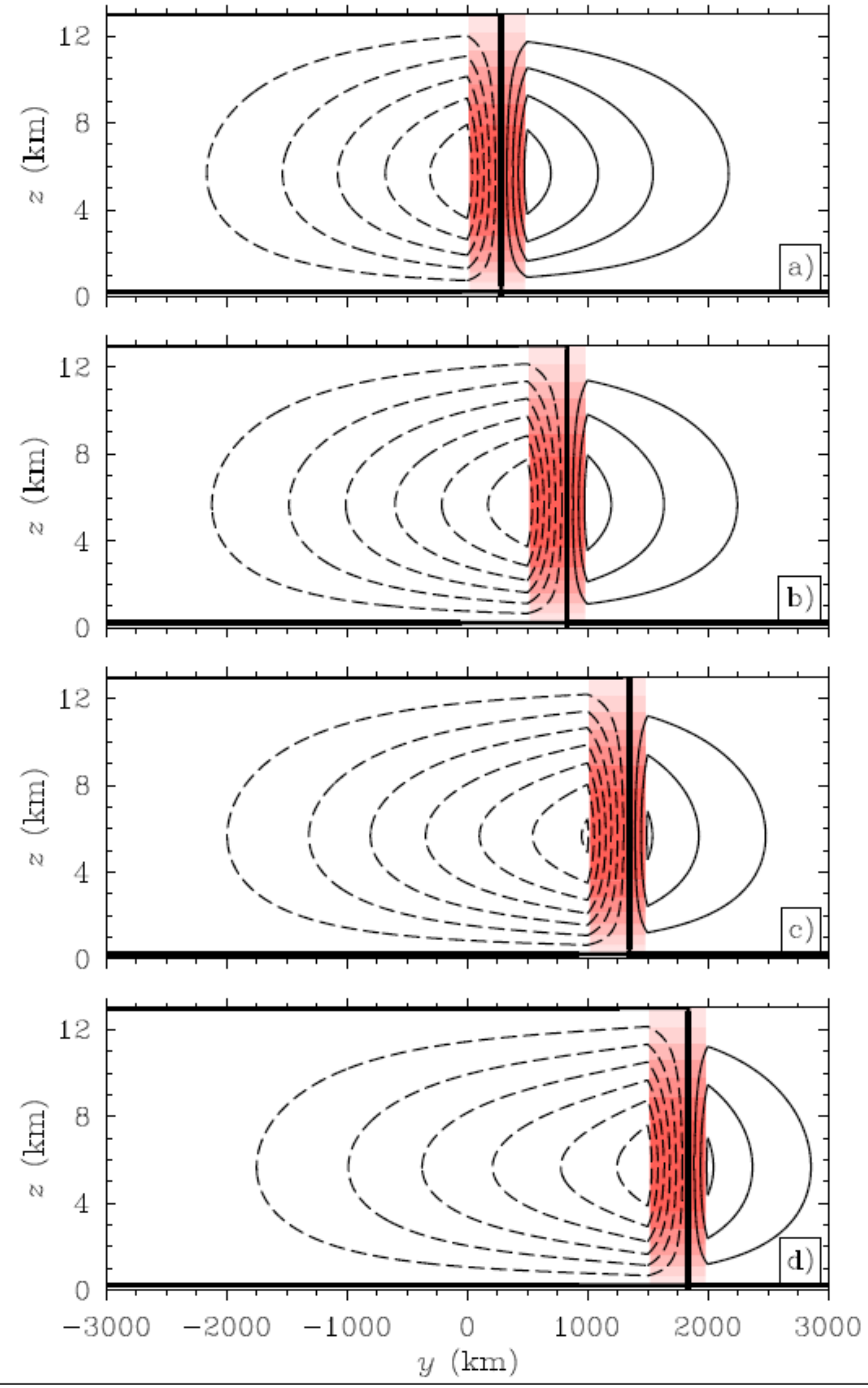
- Deep diabatic heating forcing

- Black contours: streamfunction in units of m^2/s.

- Red shading: diabatic heating in K/day.

- As ITCZ moves poleward, asymmetry between winter and summer cells increases in a), b), c).
    - Max asymmetry > 2:1 ~1200 km.

- Winter cell crosses the equator where inertial stability is smallest, less resistance to meridional motion.

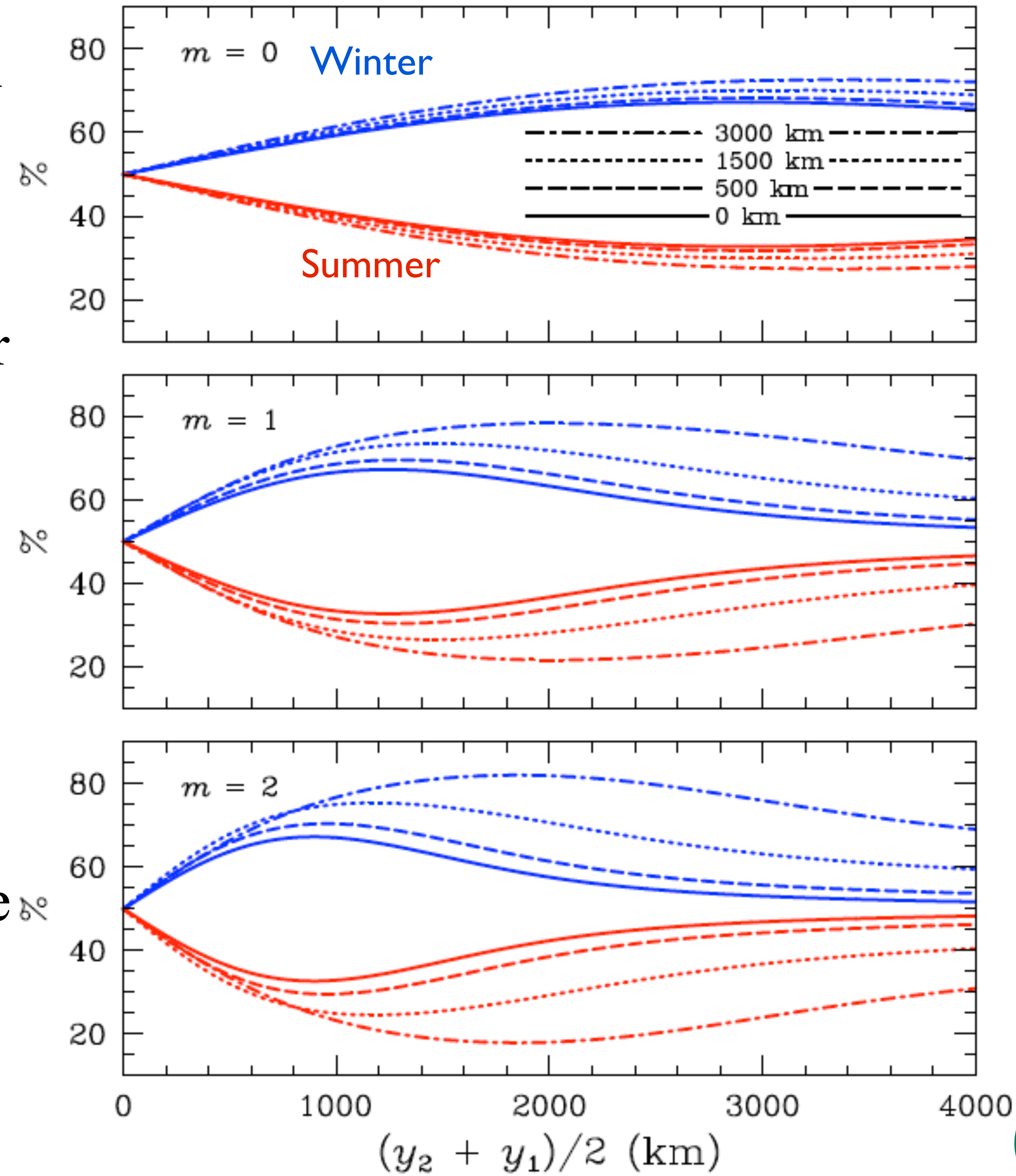


Forcing #1 - ψ Asymmetry

- Percentage of total mass flux for winter and summer HCs.

- As the ITCZ widens, the asymmetry between the winter and summer HCs increases and the latitude of max asymmetry moves poleward.

- As the vertical wavenumber increases, the asymmetry between the winter and summer HCs increases and the latitude of max asymmetry moves equatorward.

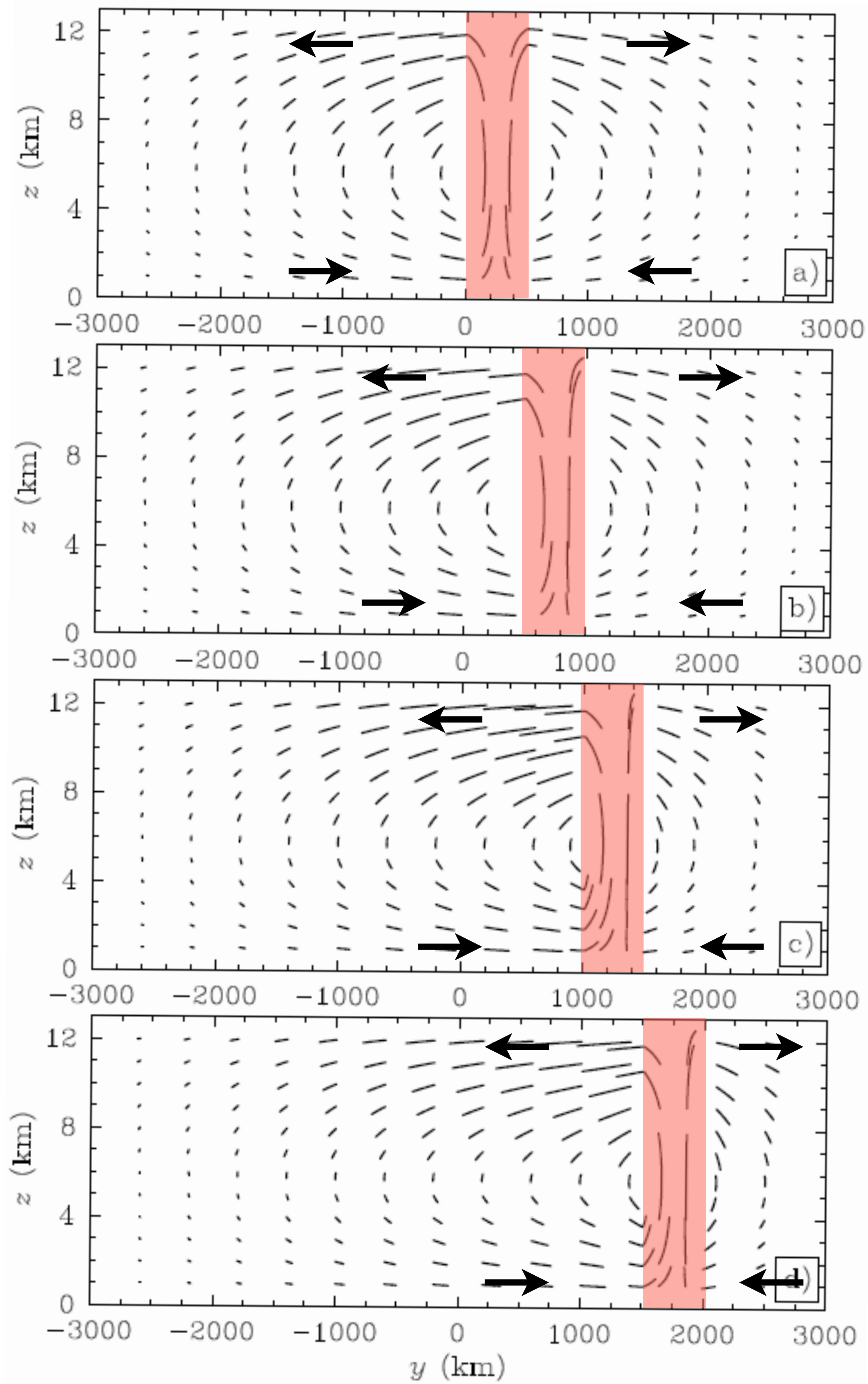


Forcing #1 - Trajectories

- Three day parcel trajectories show the winter HC experiences less resistance to horizontal motion.

- Approximate time it takes for parcels to complete one revolution in the winter cell ~1-2 months.

- **Note:** Zonal velocities are significantly larger than meridional velocities.



Forcing #2 - Shallow HC

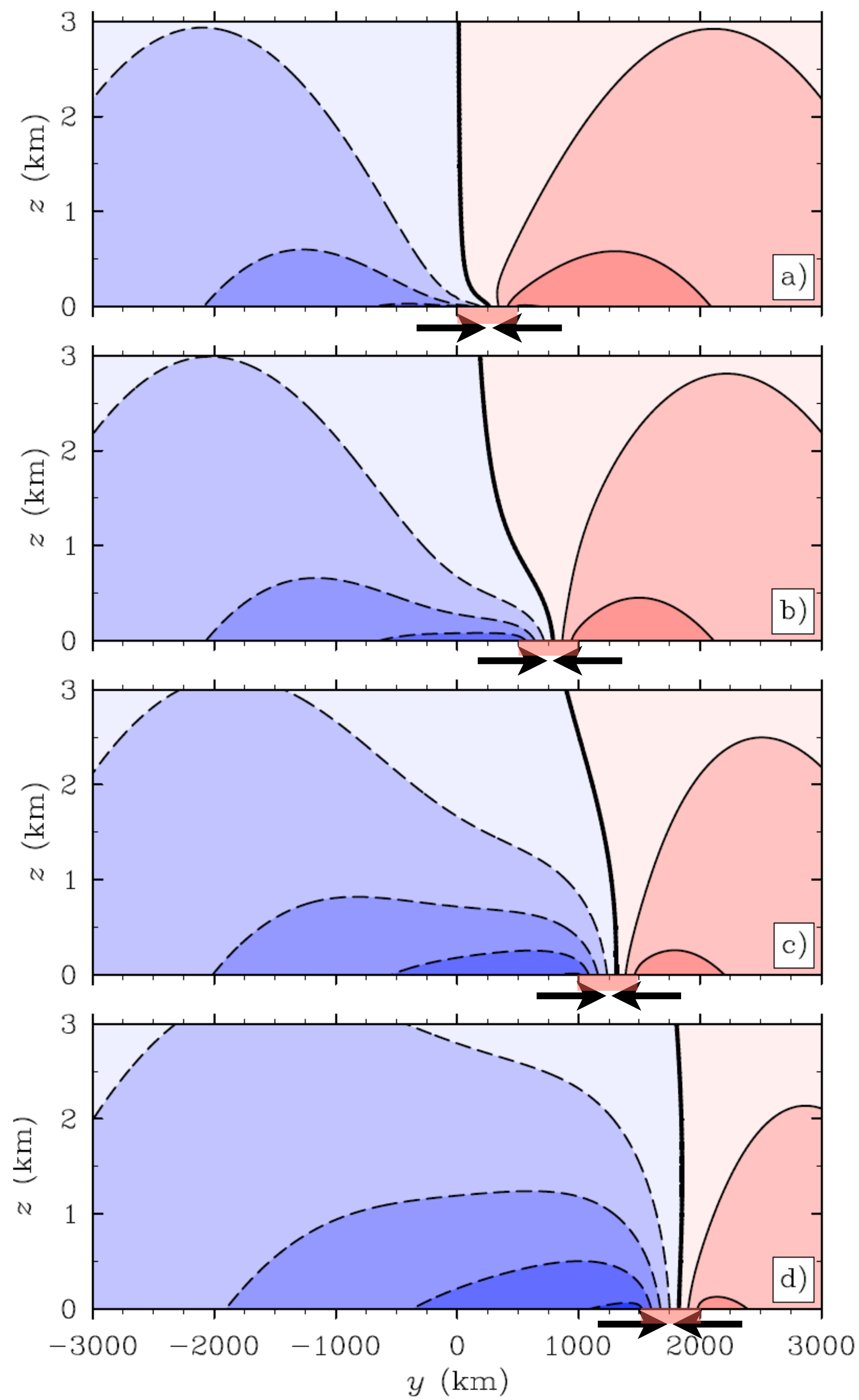
- Ekman pumping forcing only

- Contours: streamfunction, in units of m²/s.

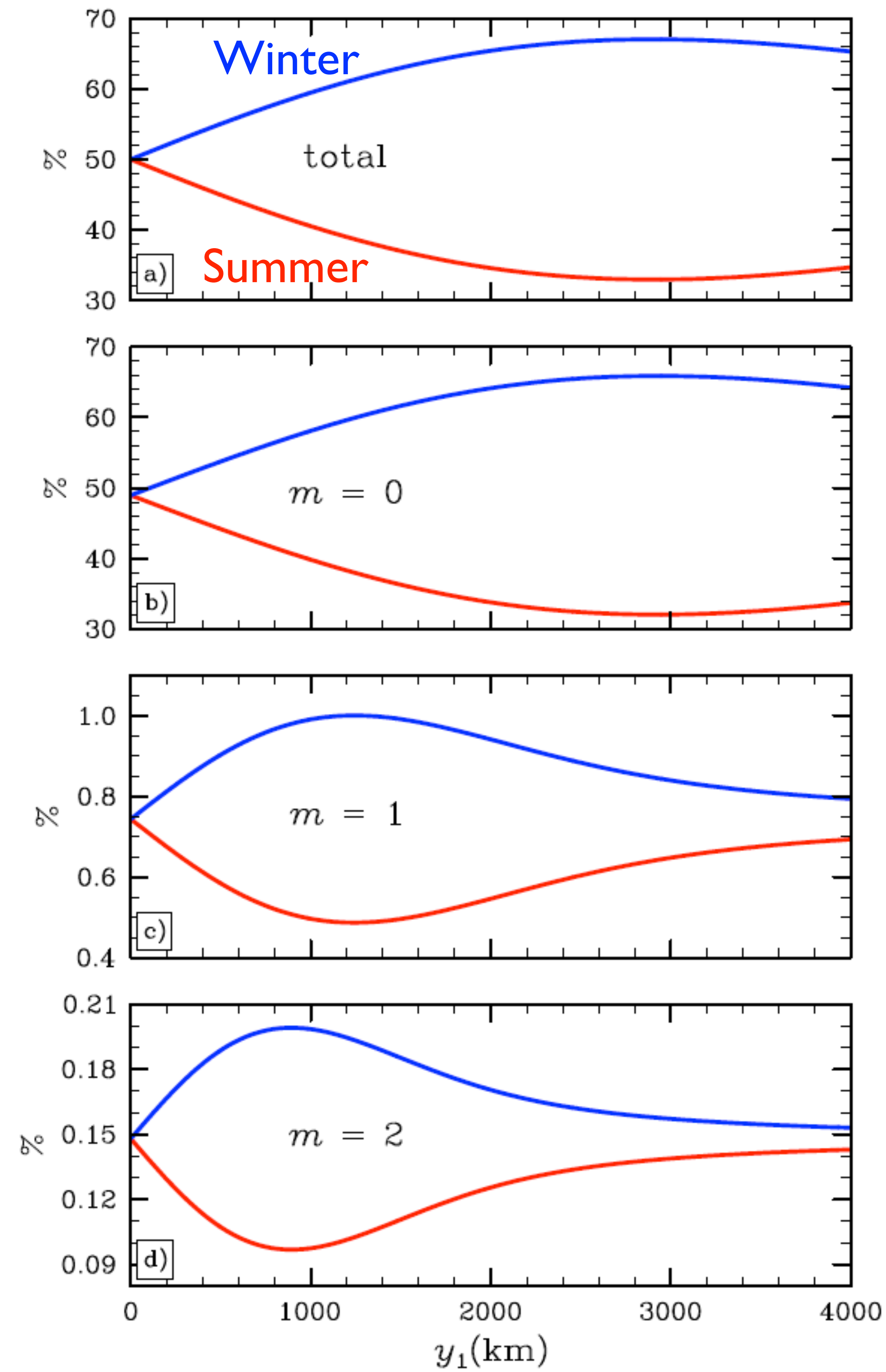
- **Note:** Vertical domain is smaller, $0 \leq z \leq 3$ km.

- As ITCZ moves poleward, asymmetry between winter and summer cells increases in a)-d). Max asymmetry ~ 2:1.

- Winter cell crosses the equator where inertial stability is smallest, less resistance to meridional motion.



Forcing #2 - ψ Asymmetry

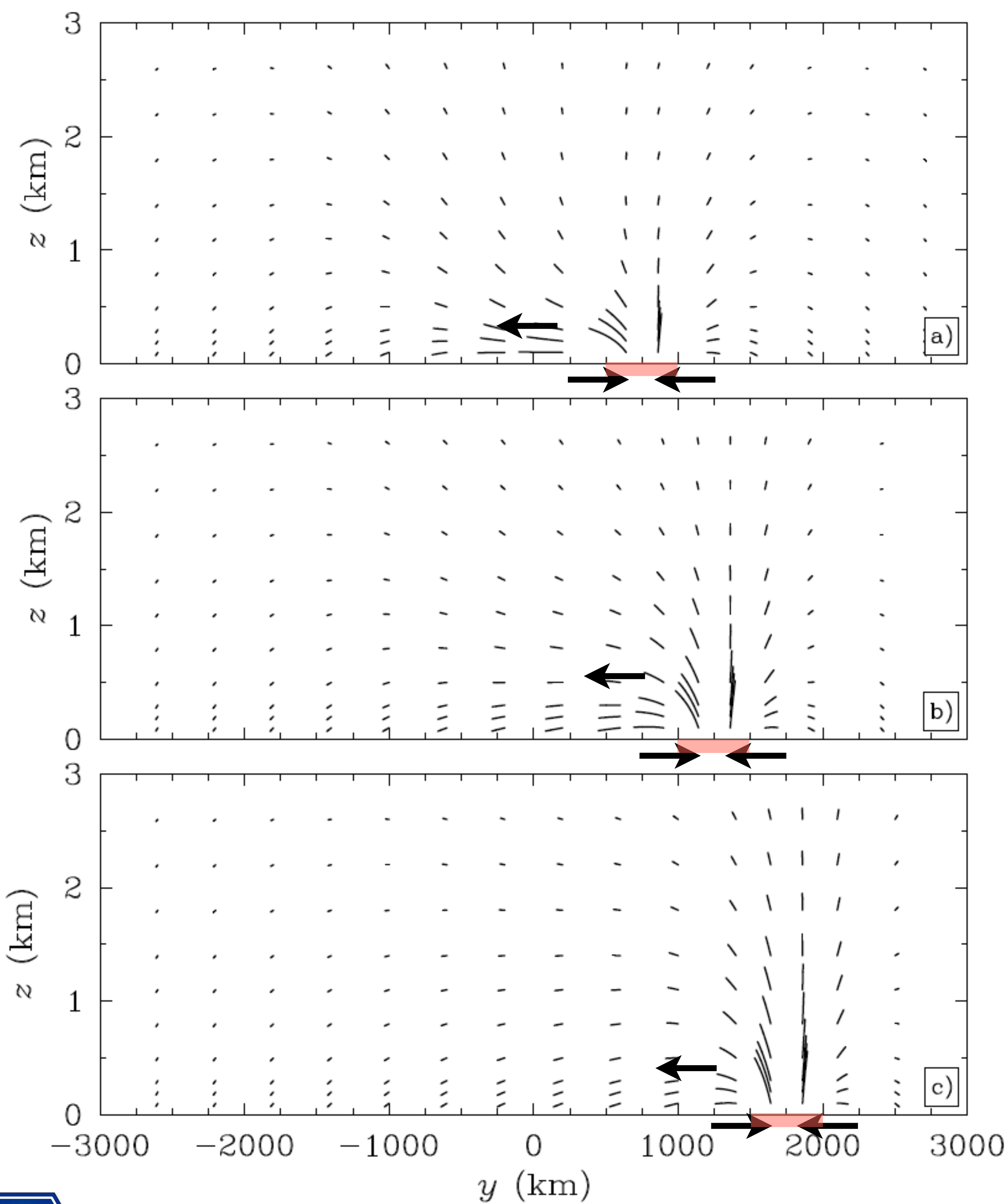


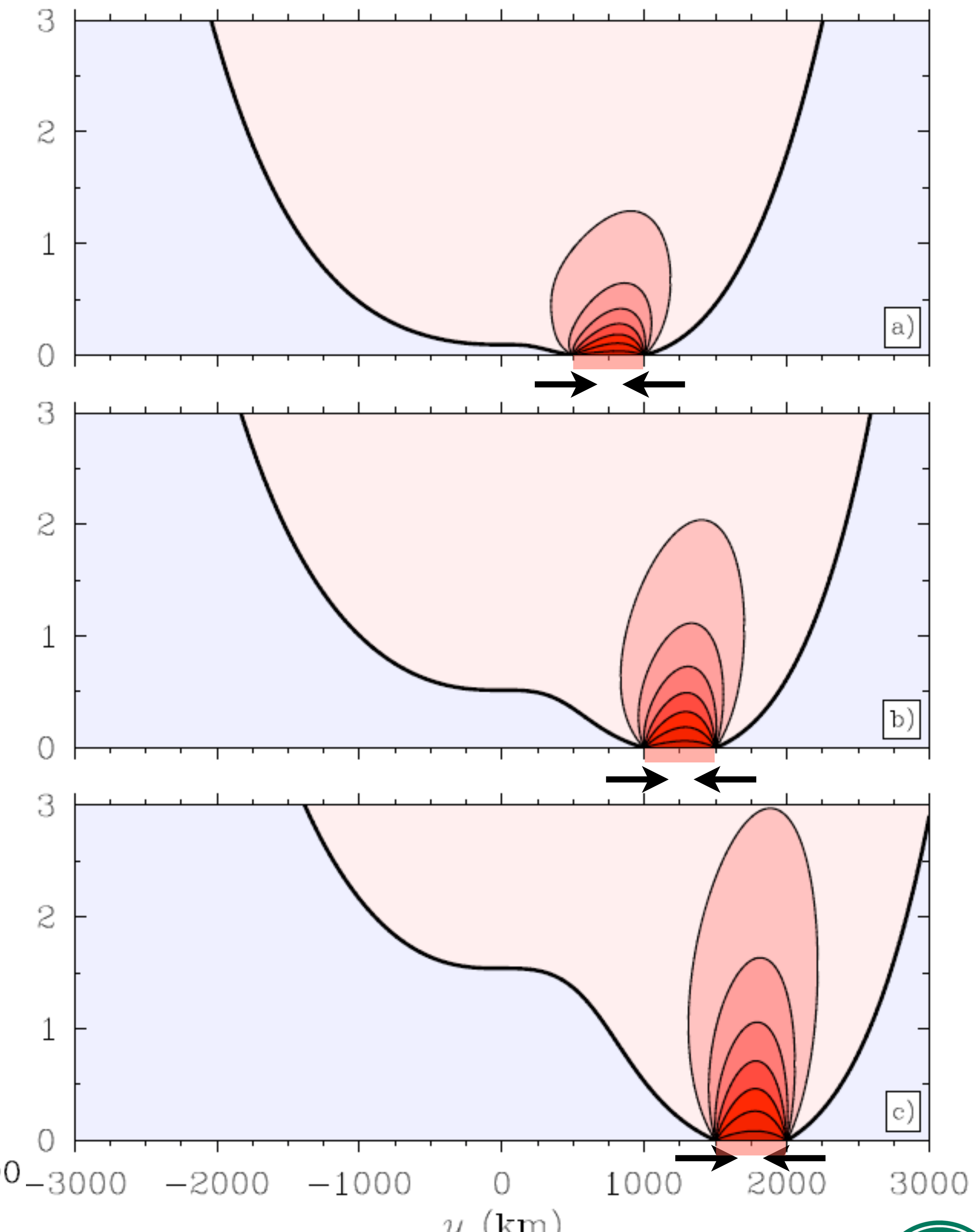
- Percentage of total mass flux for winter and summer HCs for an infinitesimally thin ITCZ.

- Maximum asymmetry occurs when ITCZ centered ~2800 km.

- Solution of mass flux is dominated by the external mode, $m = 0$.

Forcing #2 - Trajectories, w





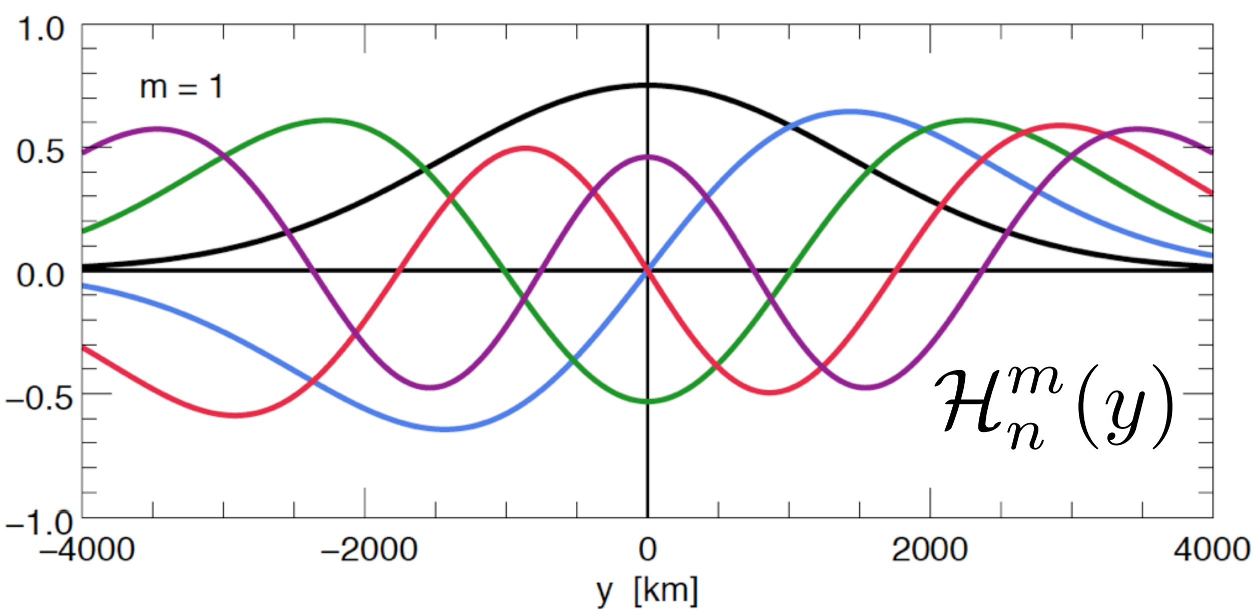
Transient Deep HC

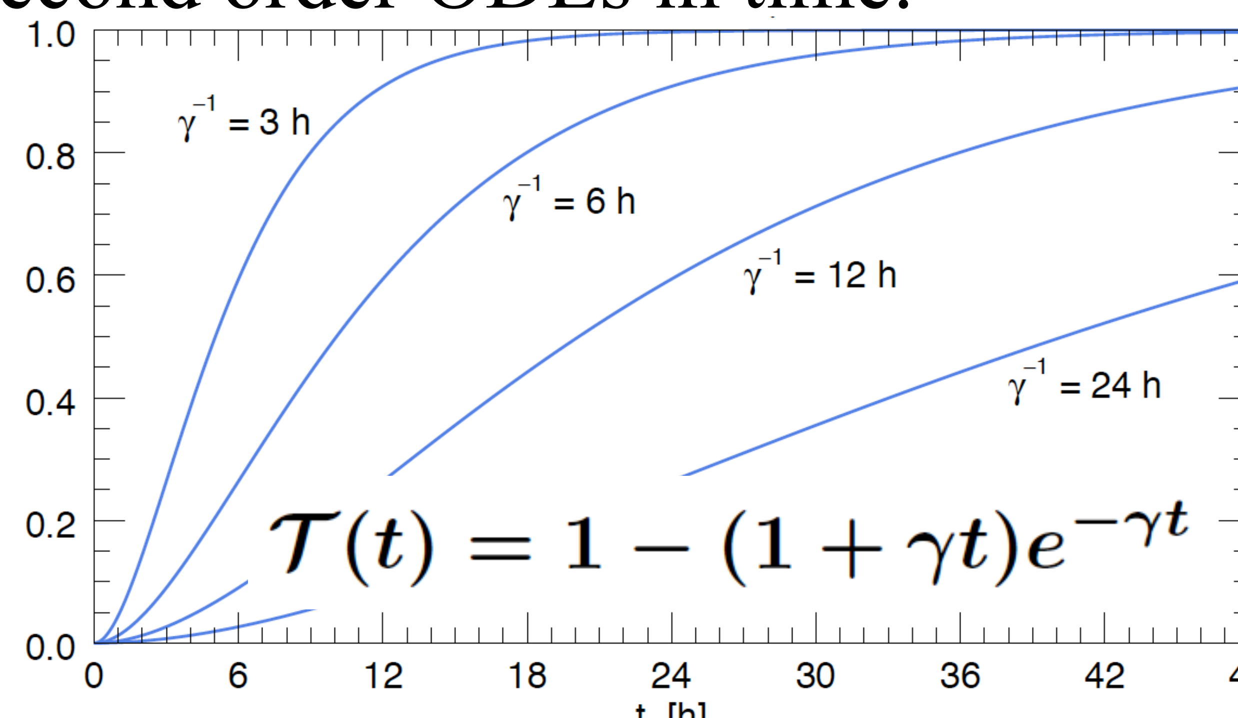
- Retain the $\partial^2/\partial t^2$ terms, solutions now contain inertia-gravity waves.

$$N^2 e^{z/H}\frac{\partial^2\psi}{\partial y^2} + \left(\frac{\partial^2}{\partial t^2} + \beta^2 y^2\right)\frac{\partial}{\partial z}\left(e^{z/H}\frac{\partial\psi}{\partial z}\right) = \frac{g}{c_p T_0}\frac{\partial Q}{\partial y}$$

- **Step 1:** Perform vertical transform (same transform as before).

- **Step 2:** Perform meridional Hermite transform of the resulting meridional structure equation to obtain a set of second order ODEs in time.





Initial Conditions: $\hat{\psi} = 0$ and $\frac{\partial\psi}{\partial t} = 0$ at $t = 0$

Transient Deep HC

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$$N^2 e^{z/H}\frac{\partial^2\psi}{\partial y^2} + \left(\frac{\partial^2}{\partial t^2} + \beta^2 y^2\right)\frac{\partial}{\partial z}\left(e^{z/H}\frac{\partial\psi}{\partial z}\right) = \frac{g}{c_p T_0}\frac{\partial Q}{\partial y}$$

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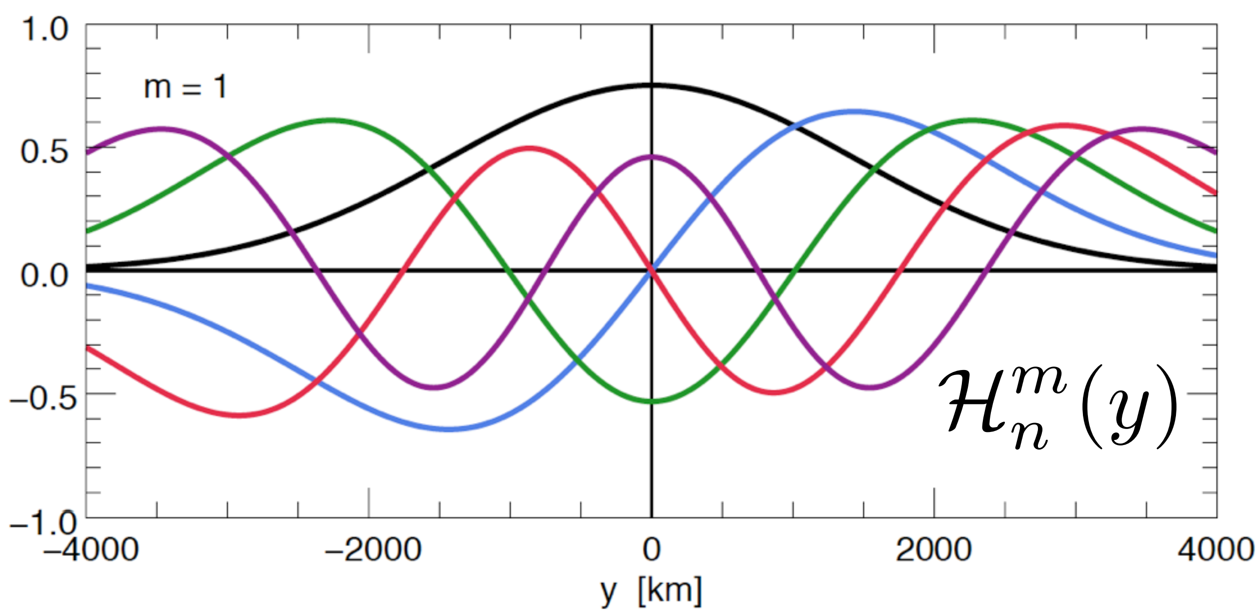
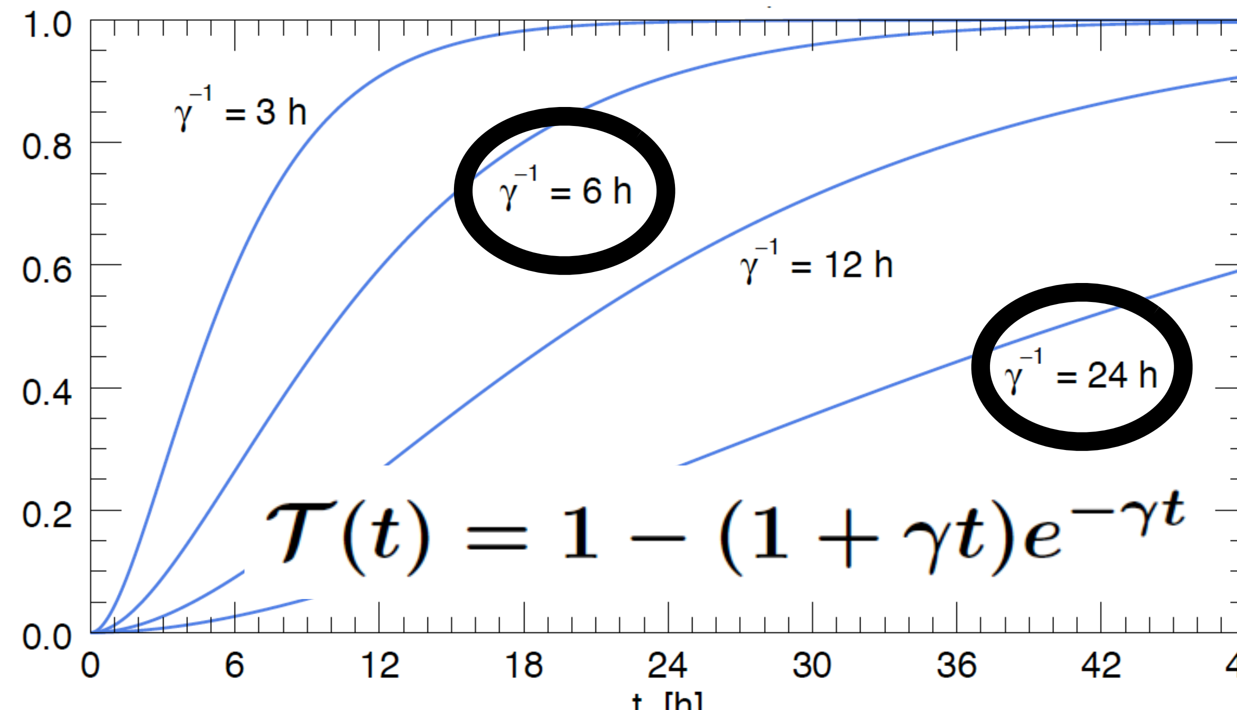




Initial Conditions: $\hat{\psi} = 0$ and $\frac{\partial\psi}{\partial t} = 0$ at $t = 0$

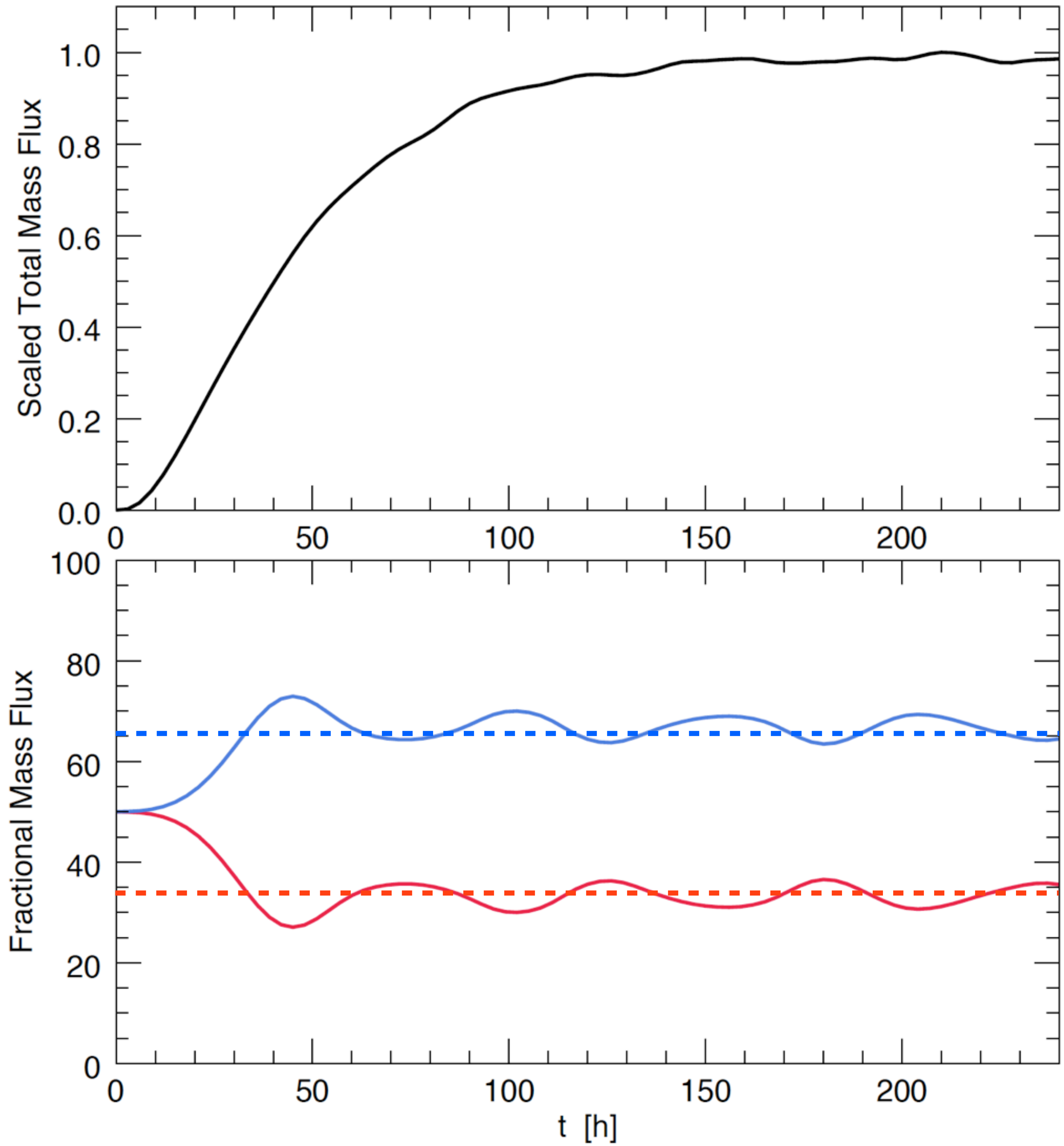
F #1 - Transient Asymmetry

ITCZ between 500-1000 km

Slow diabatic switch on (γ =(24 h)$^{-1}$)

~50-60 h inertia-gravity wave oscillations

Similar 2:1 asymmetry to balanced solutions over time



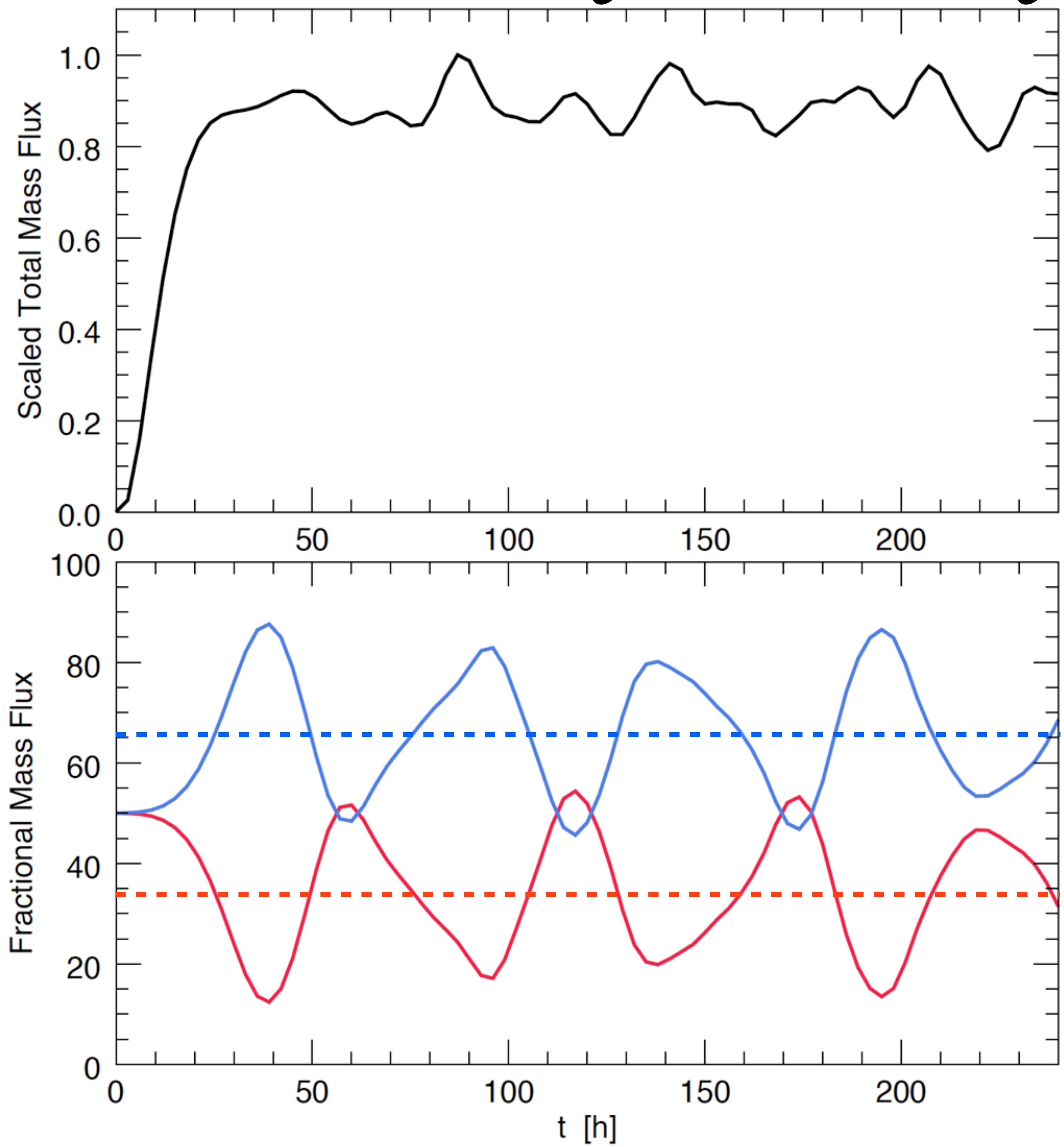
F #1 - Transient Asymmetry

ITCZ between 500-1000 km

Fast diabatic switch on ($\gamma =(6\ \mathrm{h})^{-1}$)

~40-60 h inertia-gravity wave oscillations (irregular)

Up to ~9:1 asymmetry

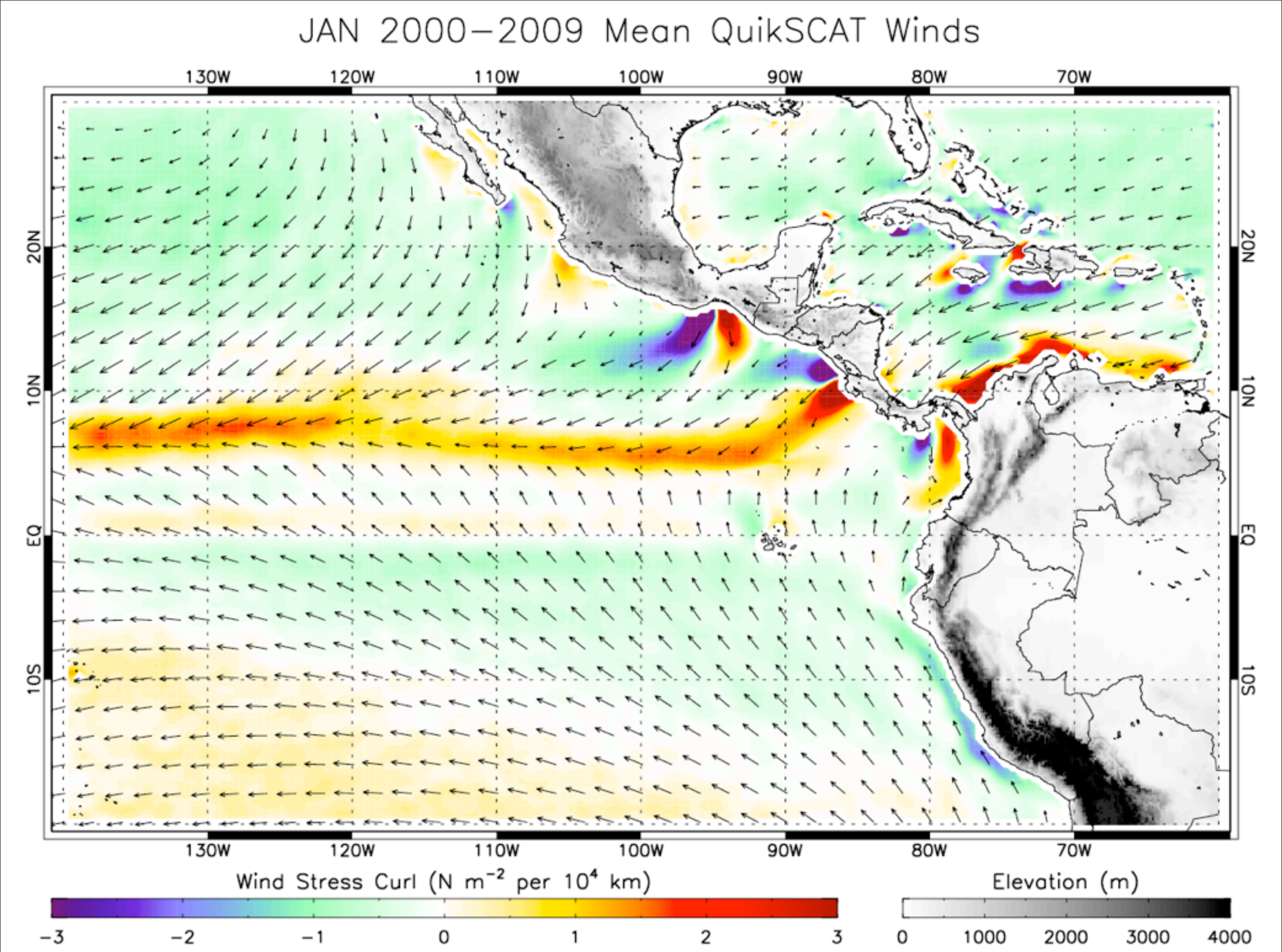


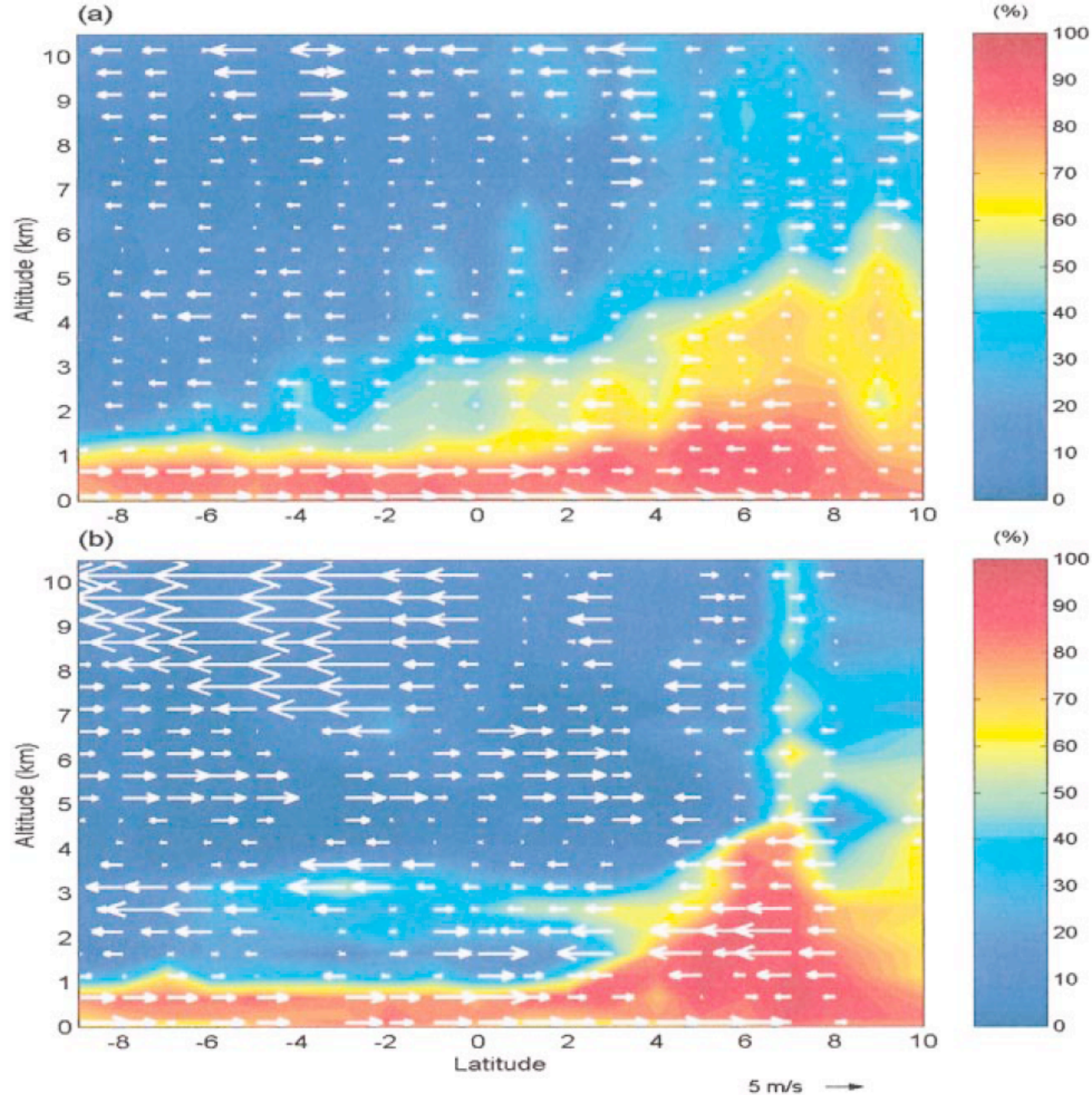
Concluding remarks

- An idealized zonally symmetric model on the equatorial β-plane was used to investigate deep and shallow circulations in the tropics.
- Prescribed ITCZ forcings: m = 1 diabatic heating and Ekman pumping at the top of the boundary layer.
- **Step 1:** Vertical transform by utilizing eigenfunctions and eigenvalues to remove z derivatives.
- **Step 2:** Use Green's functions or meridional transform (Hermite functions) to remove y derivatives.
- The balanced model illustrates there is a deep Hadley circulation when diabatic heating is present, and there is a shallow Hadley circulation in the absence of diabatic heating due to Ekman pumping.
- Analytical formulas are derived of the asymmetry between the winter and summer cells as a function of ITCZ location, width, and vertical wavenumber (due to inertial stability).

Concluding remarks

- This work has shown that Ekman pumping is a viable forcing mechanism for the shallow Hadley circulation.
- However, diabatic heating due to shallow precipitating convection and surface heating (e.g., land/sea breezes due to SST gradients, discussed in Nolan et al. 2007, 2010) are also viable forcing mechanisms.
- In fact, one could use a value for vertical velocity at the top of the boundary layer in our model due to other processes such as SST gradients and obtain similar results.
- Transient switch-on diabatic heating solutions illustrate equatorially-trapped inertia-gravity waves ring between the ITCZ and a critical latitude on a timescale of 40-60 h.
- Are there observations of ringing in the Hadley cells (e.g., *v* winds)? Also, if diabatic heating has contributions from higher vertical wavenumbers, timescale of ringing will be different than 40-50 h.


JAN 2000–2009 Mean QuikSCAT Winds
130W
120W
110W
100W
90W
80W
70W
20N
10N
EQ
10S
Wind Stress Curl (N m⁻² per 10⁴ km)
−3
−2
−1
0
1
2
3
Elevation (m)
0
1000
2000
3000
4000



(a)
(%)
Altitude (km)
0
1
2
3
4
5
6
7
8
9
10
-8
-6
-4
-2
0
2
4
6
8
10
100
90
80
70
60
50
40
30
20
10
0
(b)
(%)
Altitude (km)
Latitude
5 m/s

Lower boundary condition

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + gw = g\mathcal{W} \quad \text{at} \quad z = 0.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right) + \left(\frac{\partial^2}{\partial t^2} + \beta^2 y^2\right) v = 0,$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right) + N^2 w = \frac{g}{c_p T_0} Q.$$

$$e^{-z/H} v = -\frac{\partial \psi}{\partial z} \quad \text{and} \quad e^{-z/H} w = \frac{\partial \psi}{\partial y}.$$

$$g\frac{\partial^2 \psi}{\partial y^2} + \left(\frac{\partial^2}{\partial t^2} + \beta^2 y^2\right)\frac{\partial \psi}{\partial z} = g\frac{\partial \mathcal{W}}{\partial y} \quad \text{at} \quad z = 0.$$

Eigenvalues, h_m

- The eigenvalues are also equivalent depths h_m, and can be expressed as internal gravity wave speeds $(gh_m)^{1/2}$, Rossby lengths b_m, and Lamb's parameters ϵ_m.

- One external mode $m = 0$, and many internal modes $m = 1, 2, \ldots \infty$.

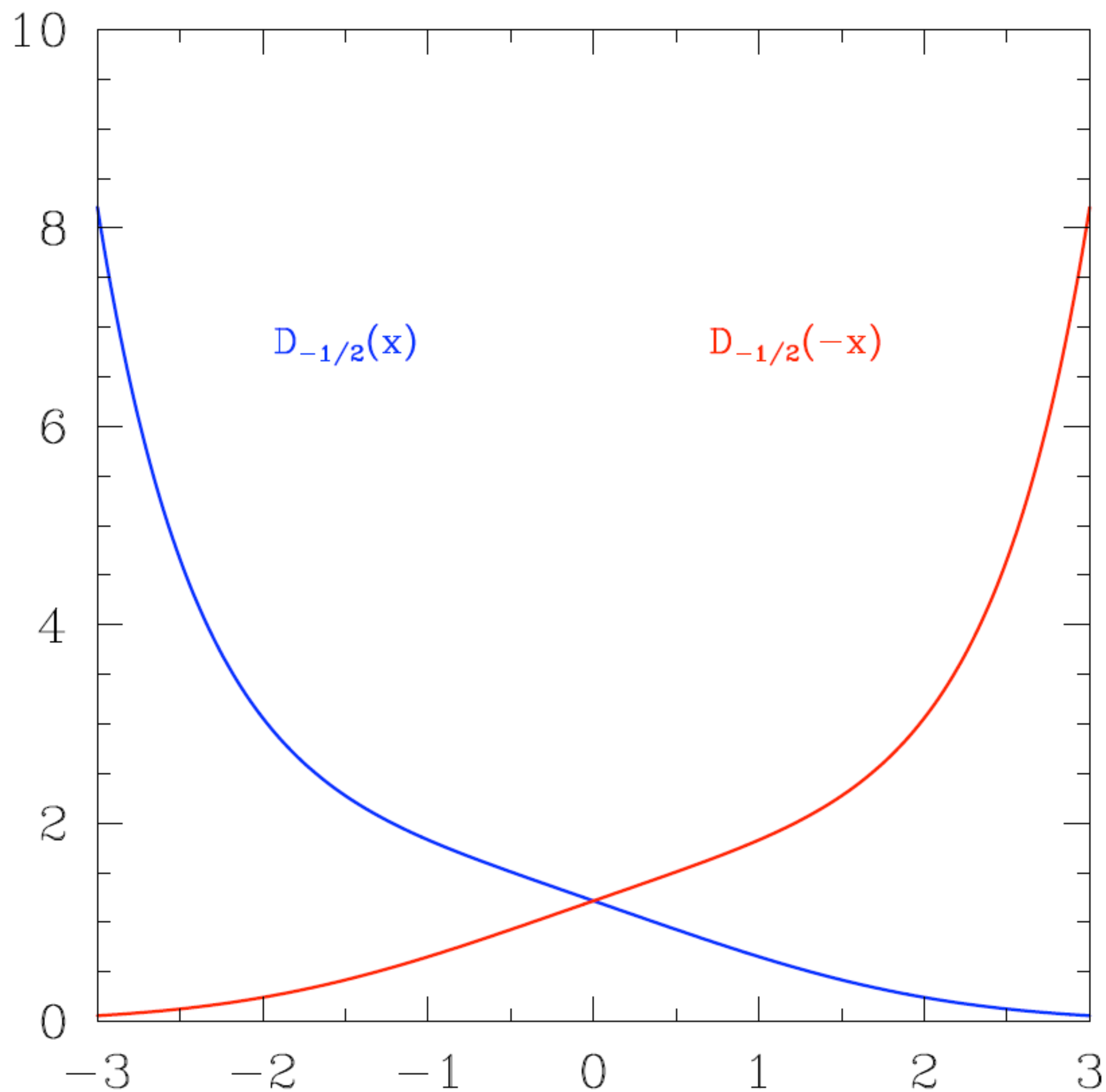
m	h_m (m)	$(gh_m)^{1/2}$ (m s^{-1})	b_m (km)	ϵ_m
0	7099	263.8 (—)	2400	12.41
1	229.8	47.46 (48.27)	1018	383.4
2	61.42	24.53 (24.65)	732.0	1434

- Diabatic heating uses only first internal mode.

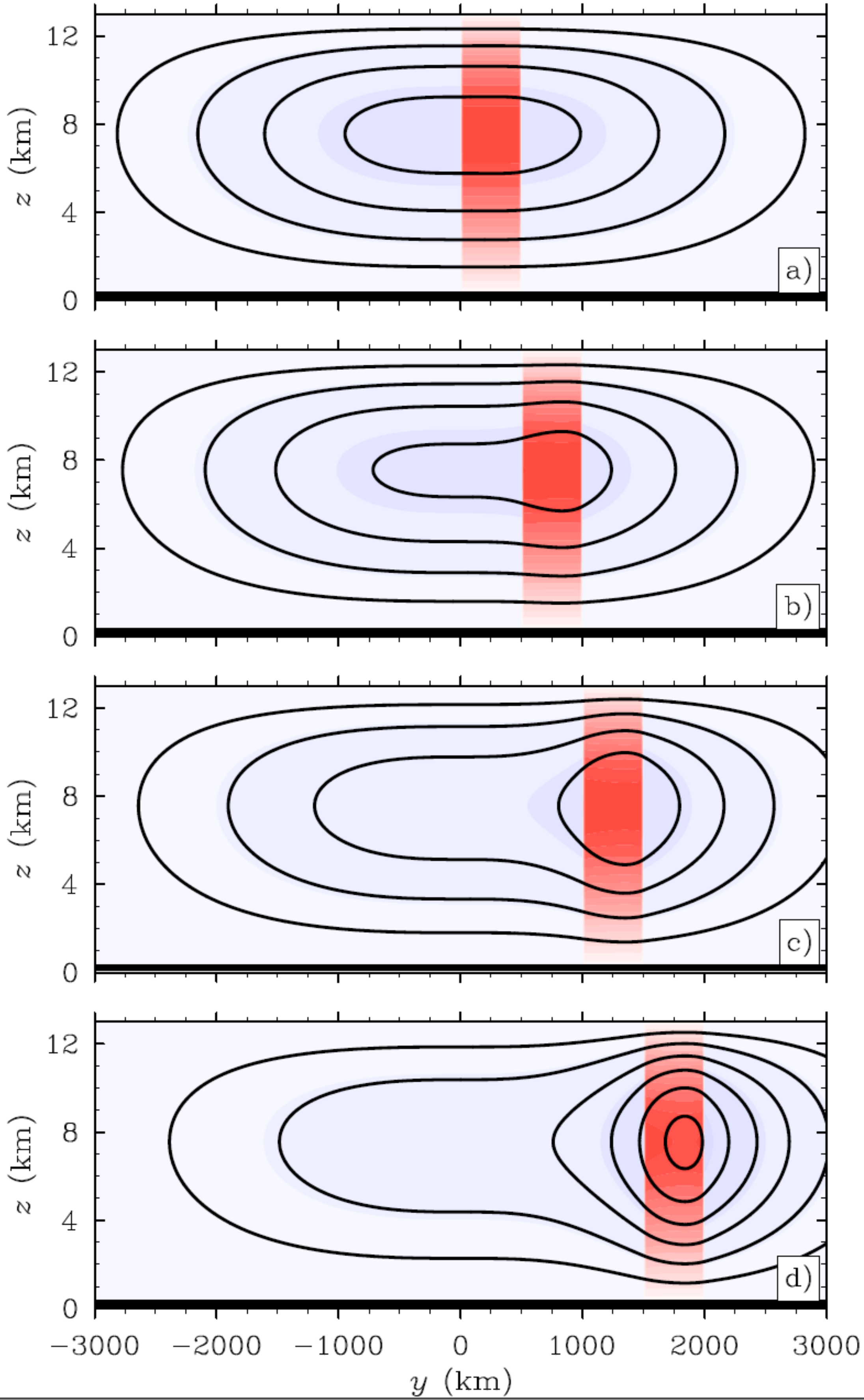
- Ekman pumping uses the external mode and $O(100)$ of internal modes.

Parabolic Cylinder Functions

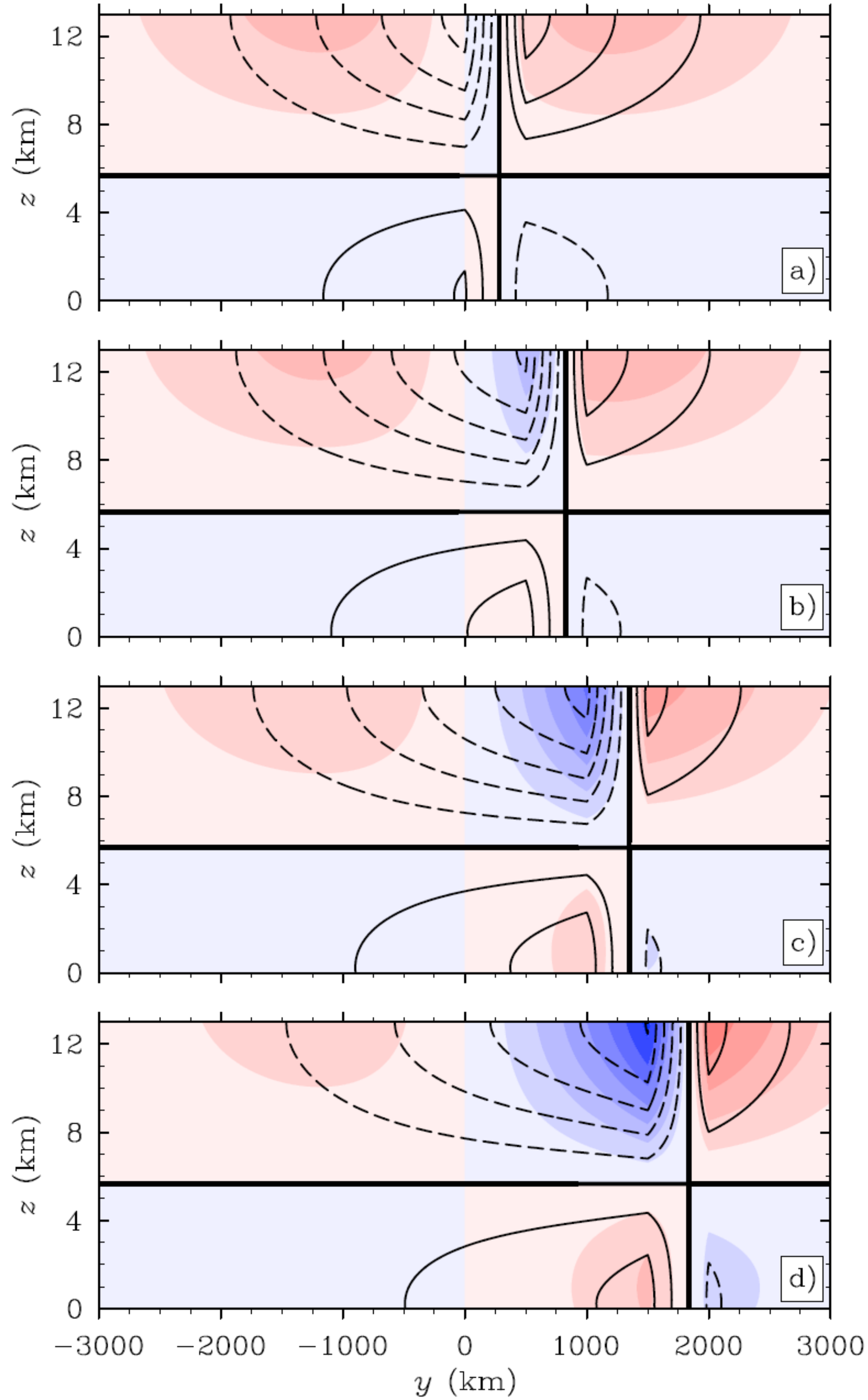
$$\frac{d^2 D_\nu}{dx^2} + \left(\nu + \frac{1}{2} - \frac{1}{4}x^2\right) D_\nu = 0$$



F #1 - T_t, w fields



F #1 - u_t, v fields



F #2 - Boundary Layer Equ.

$$\cancelto{}{\frac{\partial u_b}{\partial t}} - \beta y v_b = -k u_b,$$

$$\cancelto{}{\frac{\partial v_b}{\partial t}} + \beta y u_b = -k v_b + \beta y u_g,$$

$$-h_E \frac{\partial v_b}{\partial y} = w(y, 0, t) - w(y, -h_E, t) = \mathcal{W}(y, t),$$

$$\beta y \, u_g = -\frac{\partial \phi}{\partial y}.$$

$$u_b(y, t) = \left(\frac{\beta^2 y^2}{k^2 + \beta^2 y^2} \right) u_g(y, t),$$

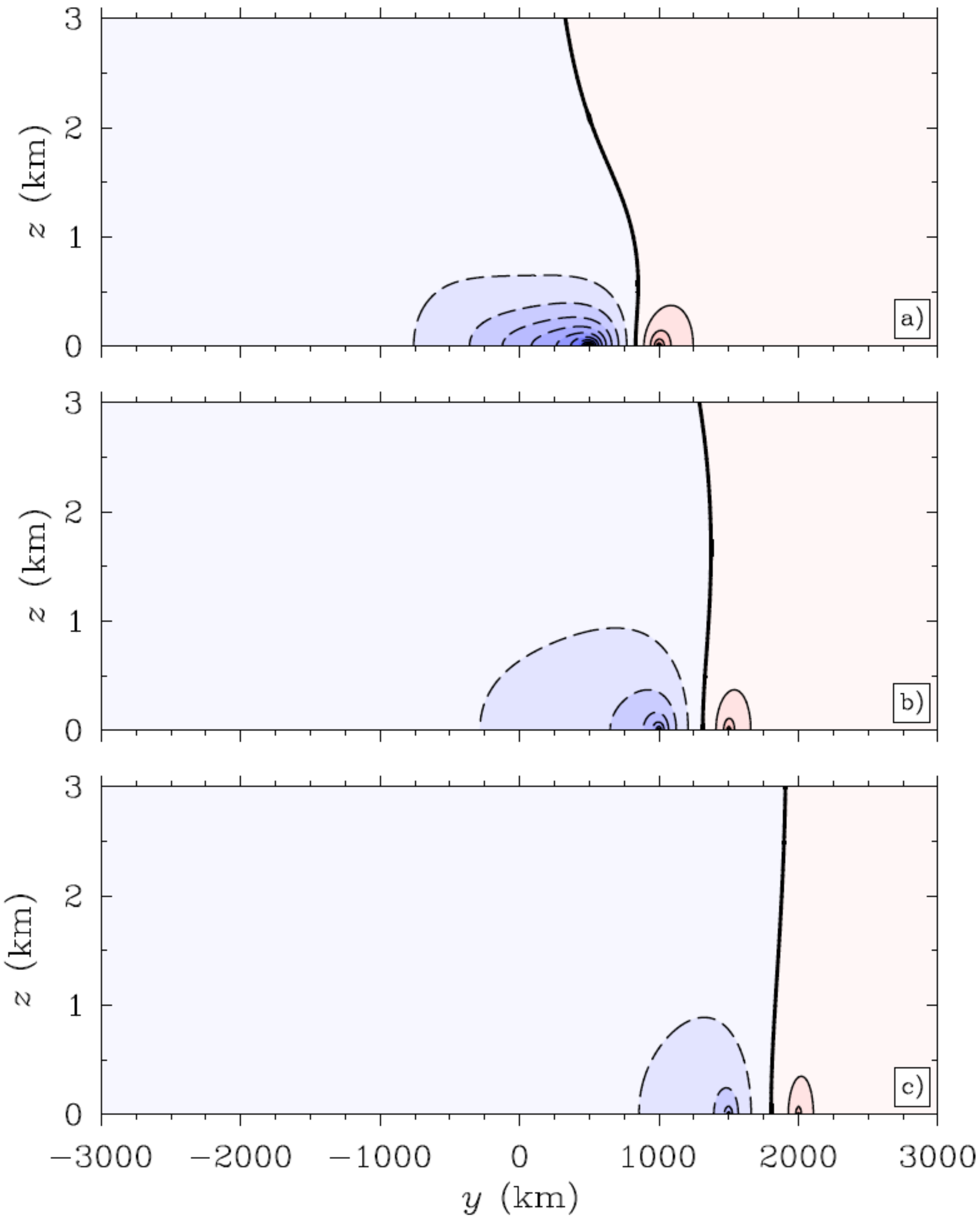
$$v_b(y, t) = \left(\frac{k \beta y}{k^2 + \beta^2 y^2} \right) u_g(y, t).$$

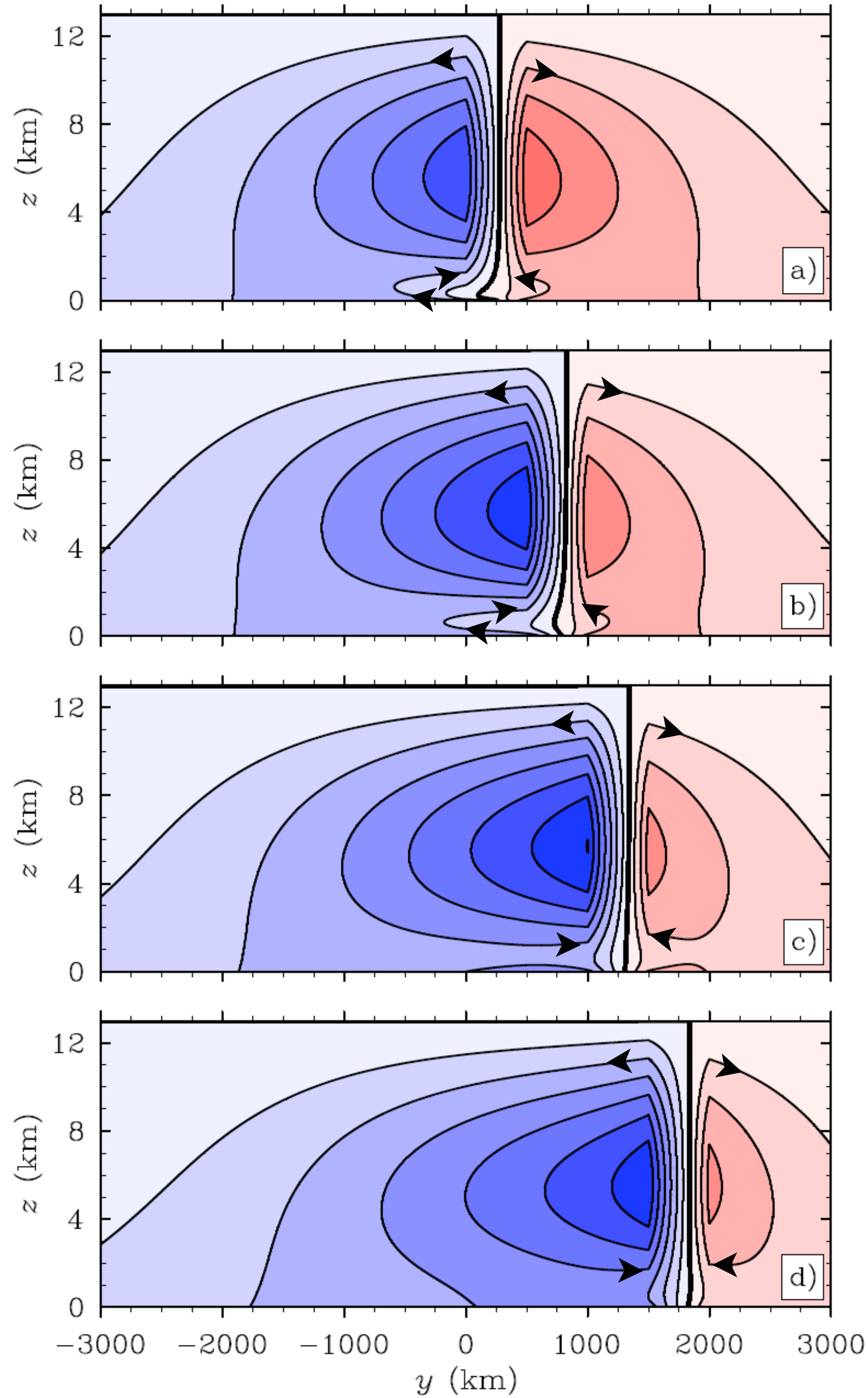
$$u_g(y_1) = 3 \text{ m/s}$$

$$u_g(y_2) = -3 \text{ m/s}$$

$$\mathcal{W}_{\text{ave}} = 4 \text{ mm/s}$$

F #2 - v field

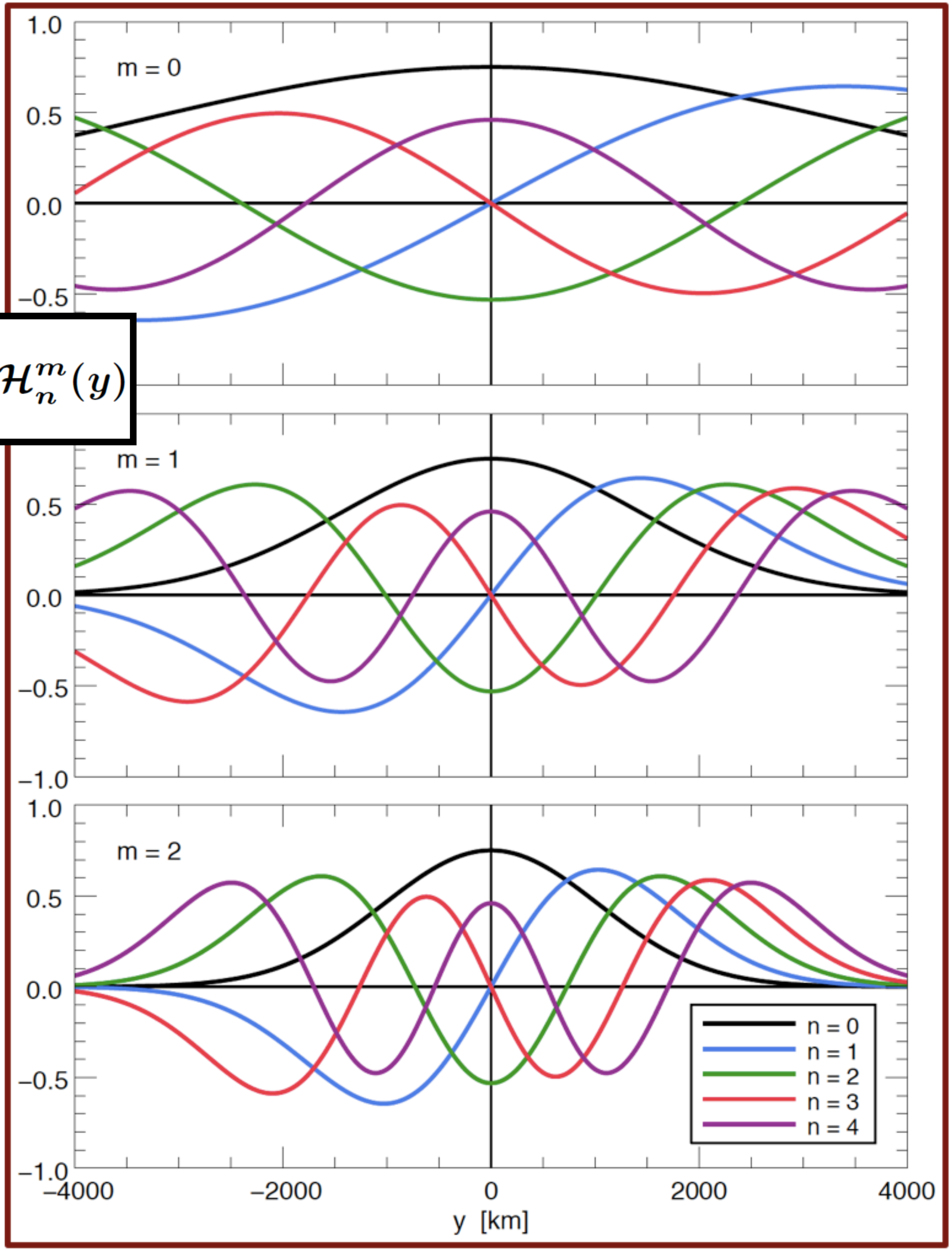


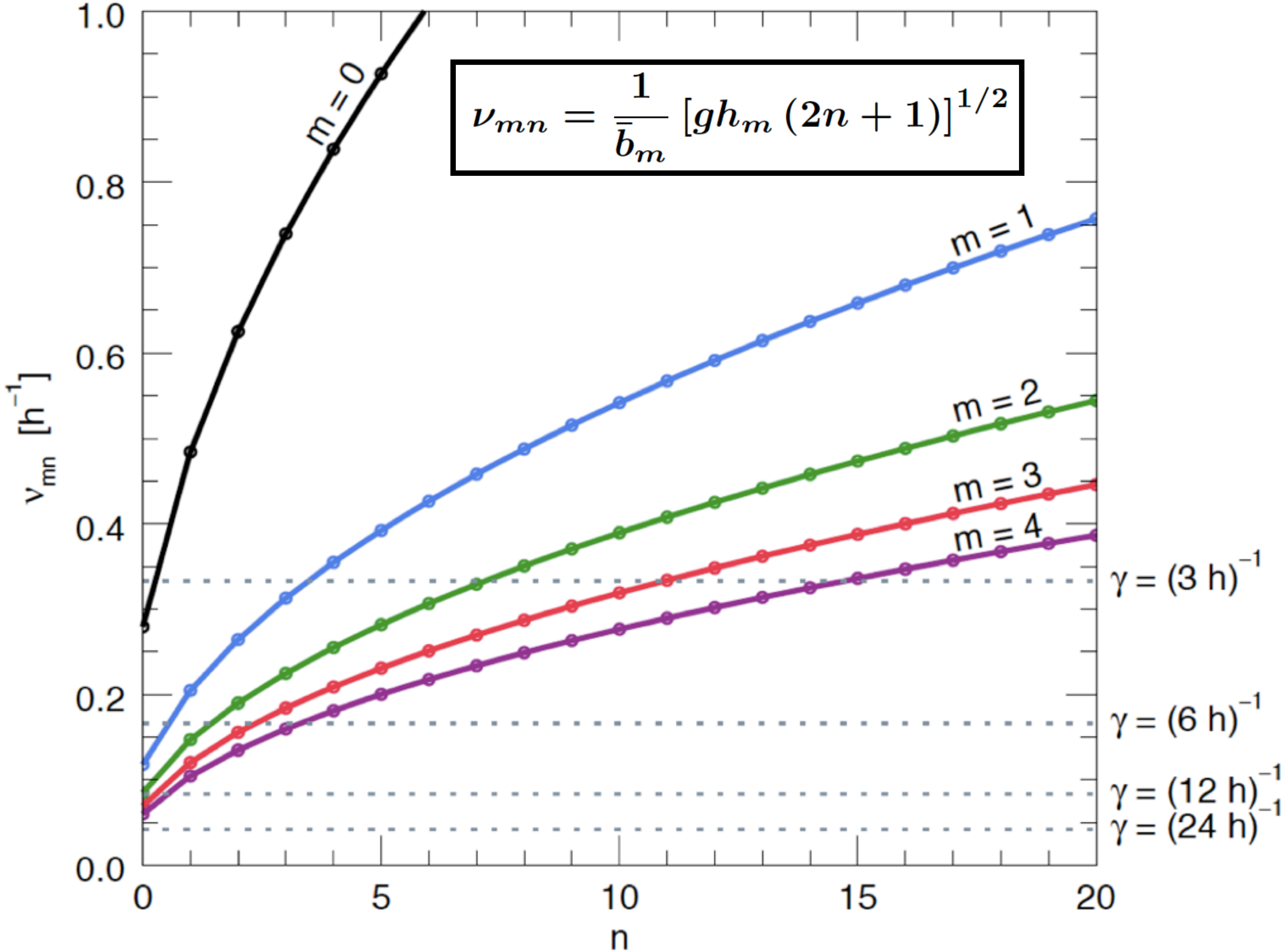


12
8
4
0
z (km)
a)
b)
c)
d)
−3000
−2000
−1000
0
1000
2000
3000
y (km)

Eigenfunctions:

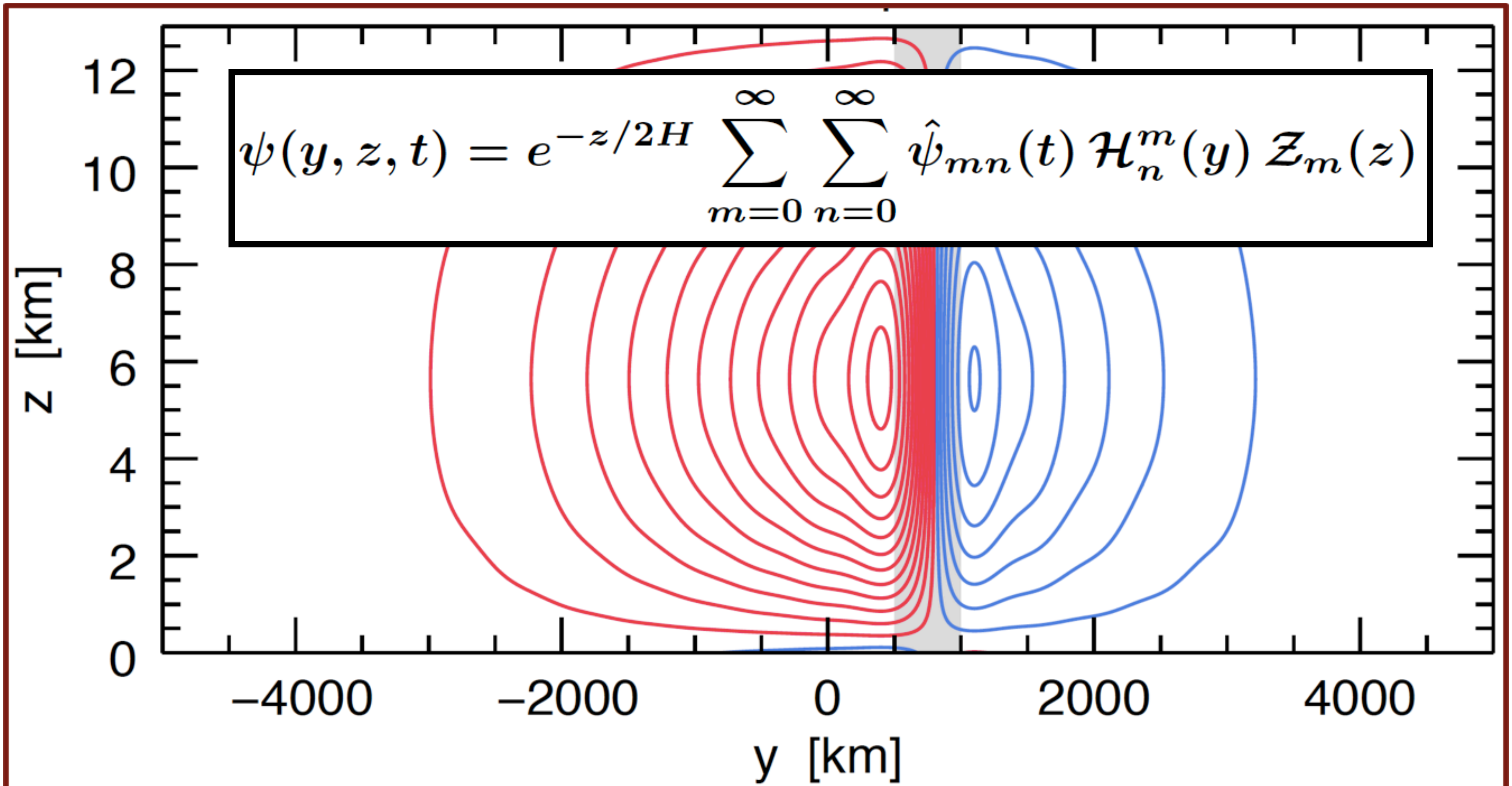
$$\left(\frac{d^2}{dy^2} - \frac{y^2}{\bar{b}_m^4}\right)\mathcal{H}_n^m(y) = -\left(\frac{2n+1}{\bar{b}_m^2}\right)\mathcal{H}_n^m(y)$$





$$\nu_{mn} = \frac{1}{\bar{b}_m}\left[g h_m (2n+1)\right]^{1/2}$$

$$
\hat{\psi}_{mn}(t) = -\frac{g h_m \mathcal{F}_{mn}}{\nu_{mn}^2}\left\{ \left(\frac{(\nu_{mn}^2 - \gamma^2)\gamma^2}{(\nu_{mn}^2 + \gamma^2)^2} \right) \cos(\nu_{mn} t) - \left(\frac{2\gamma^3 \nu_{mn}}{(\nu_{mn}^2 + \gamma^2)^2} \right) \sin(\nu_{mn} t) + 1 - \left(\frac{\nu_{mn}^2 + 3\gamma^2}{\nu_{mn}^2 + \gamma^2} + \gamma t \right) \left(\frac{\nu_{mn}^2 e^{-\gamma t}}{\nu_{mn}^2 + \gamma^2} \right) \right\}
$$



$$
\psi(y,z,t) = e^{-z/2H} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \hat{\psi}_{mn}(t)\, \mathcal{H}_n^m(y)\, \mathcal{Z}_m(z)
$$